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Foreword

This book deals with the new program of the eighth year (intermediate level) with a new spirit : the individual construction of notions, the formation of the student in order to reach communication and critical reasoning, the preservation of the link between mathematics and situations of our daily life.

Each chapter is formed of several parts :

- **The preliminary activities.** It is short and accessible, allowing the introduction of a new notion or sometimes of a part of a lesson.

- **The course.** It is clear, simple and concise; it also respects the spirit of the new curriculum. Some results are stressed in a way or another, to allow the student to refer to them.

- **The exercises and the problems.** We have suggested a number of exercises; they are chosen in an interesting way and are presented in **three headings.**

  – **Test the knowledge.** This part consists of the most accessible exercises, allowing to check the minimal acquisition of the student.

  – **For seeking.** It consists of exercises which demand a supplementary effort of reflexion.

  – **The test.** The goal of this part is to test if the student has assimilated well the studied notion.

In addition to this, a great number of problems are chosen from real life, studying the real, familiar but not strange situations for students. This is done to reach a determined goal, which is to consolidate the link between daily life and mathematics.

We wish the usage of calculators, especially that this is required in the new curriculum.

We wish that this work would be useful for the students of the 8th year and would contribute to improve the level of teaching mathematics.

We would be pleased and thankful to receive all suggestions, criticism or advice.

The authors

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1 POWERS

Objectives

1. Perform operations on powers having natural numbers as exponents.
2. Using powers of 10 having integers as exponents.

CHAPTER PLAN

COURSE

COURSE

1 DEFINITION

a is an integer and n is a natural number greater than 1 : **the product of n factors all equal to** a is written a^n.

$$a^n = \underbrace{a \times a \times \dots a}_{n \text{ factors}}$$

a^n is called the **n**

a^n is called the **n^{th} power of** a.
It is read «**a to the n^{th} power**» or «**a exponent n**».
a is the base and n is **the exponent** of this power.
We also write $a^1 = a$.
For any number a different than zero, $a^0 = 1$.

EXAMPLES

- $(-2)^3$, $(-2)^4$ and $(-2)^{13}$ are powers of -2 .
- $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$.
- $(-2)^{13}$ is the thirteenth power of -2 .
- $(-3)^3 = (-3) \times (-3) \times (-3) = -27$.

Application 1

Calculate : ⊙ 4^3 and $(-4)^3$ ⊙ $(-1)^2$; $(-1)^3$ and $(-1)^5$.

2 USING THE CALCULATOR (*fx*95*MS* - *fx*100*MS*)

In the following chart, you can find the patterns that help you to calculate the powers represented in the first column, using a scientific calculator.

4^3	4 ^ 3 = 64
$(-2)^3$	(– 2) ^ 3 = –8
$\left(\frac{3}{4}\right)^4$	(3 $a^{b/c}$ 4) ^ 4 = $\frac{81}{256}$
$(-4)^2 \times 2^5$	(– 4) ^ 2 × 2 ^ 5 = 512
$[(-2)^3]^4$	(– 2) ^ 3 ^ 4 = 4096

Application 2

Calculate using the calculator : 5^4 ; $(2^4)^3$; $(-6)^5$; $(-2.8)^4$; $\dfrac{4^2}{(-2)^3}$.

3 PROPERTIES

Property 1

a is an integer, n and m are two natural numbers that are different from zero :

$$a^n \times a^m = \underbrace{\underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}} \times \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}}_{(n+m) \text{ factors}} = a^{n+m} .$$

Therefore : $\boxed{\boldsymbol{a^n \times a^m = a^{n+m}}}$

Application 3

Write in the form of a single power.

1º) $(-3)^4 \times (-3)^2$. **2º)** $\left(\frac{1}{3}\right)^5 \times \left(\frac{1}{3}\right)^6$. **3º)** $b^4 \times b^{10}$.

Property 2

a and b are two integers, n is a natural number different from zero :

$$a^n \times b^n = \underbrace{(a \times \ldots \times a)}_{n \text{ factors}} \times \underbrace{(b \times \ldots \times b)}_{n \text{ factors}} = \underbrace{(ab) \times \ldots \times (ab)}_{n \text{ factors}}$$

Therefore : $\boxed{\boldsymbol{a^n \times b^n = (a \times b)^n}}$

Application 4

Write in the form of a single power.

1º) $(-3)^5 \times (-2)^5$. **2º)** $x^5 \times y^5$. **3º)** $(-2.5)^2 \times (-4)^2$.

Property
3

a and b are two integers where $b \neq 0$ and n is a natural number different from zero.

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \ldots \times \frac{a}{b}}_{n \text{ factors}} = \frac{\overbrace{a \times \ldots \times a}^{n \text{ factors}}}{\underbrace{b \times \ldots \times b}_{n \text{ factors}}} = \frac{a^n}{b^n}\ .$$

Therefore : $\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$

Application 5

Compare $\frac{15^3}{20^3}$ and $\left(\frac{3}{4}\right)^3$.

Property
4

a is an integer, n and m are two natural numbers different from zero :

$$(a^n)^m = \underbrace{\overbrace{(a \times \ldots \times a)}^{n \text{ factors}} \times \ldots \times \overbrace{(a \times \ldots \times a)}^{m \text{ factors}}}_{(n \times m) \text{ factors}} = a^{n \times m}$$

Therefore : $\boxed{(a^n)^m = a^{n \times m}}$

Application 6

Write in the form of a single power.

1º) $[(-2)^3]^6$. **2º)** $(4^4)^3$. **3º)** $(a^3)^5$.

Property
5

a is a non zero integer, n and m are two natural numbers different from zero with $n \geq m$:

$$\frac{a^n}{a^m} = \frac{\overbrace{a \times a \times \ldots \times a}^{n \text{ factors}}}{\underbrace{a \times a \times \ldots \times a}_{\text{factors}}}\ .$$

We simplify by m equal factors of a, and we obtain :

$\boxed{\frac{a^n}{a^m} = a^{n-m}}$ or $\boxed{a^n \div a^m = a^{n-m}}$

We say that **a^n is divisible by a^m .**

Application 7

Write in the form of a single power.

1°) $\dfrac{(-5)^{13}}{(-5)^{8}}$. 2°) $\left(\dfrac{-10}{3}\right)^{14} \div \left(\dfrac{-10}{3}\right)^{9}$. 3°) $\dfrac{x^5}{x^3}$. 4°) $\dfrac{c^{10}}{c^5}$.

Property 6

a is an integer, n is a natural number :

- If $\boldsymbol{a > 0}$, then $\boldsymbol{a^n > 0}$,
- If $\boldsymbol{a < 0}$, then : $\begin{cases} \boldsymbol{a^n > 0 \text{ if } n \text{ is even}} \\ \boldsymbol{a^n < 0 \text{ if } n \text{ is odd.}} \end{cases}$

EXAMPLES

- $2^5 > 0$. - $(-2)^4 > 0$. - $(-2)^5 < 0$.

Application 8

What is the sign of : $(-3)^{15}$? $(-2.8)^8$? $(-14)^{22} \times (-103)^{17}$?

4 POWER OF 10 HAVING A POSITIVE EXPONENT : 10^n (n natural number)

Observe the following chart.

Number	Number of zeros	Writing in the form of a power of 10	Exponent of 10
10	1	10^1	1
100	2	10^2	2
1	0	10^0	0
10 000	4	10^4	4
100 000 000	8	10^8	8

For any natural number n, we have : $\boldsymbol{10^n = 1\underbrace{00 \ldots 0}_{n \text{ zeros}}}$

(n zeros to the right of 1).

5 POWER OF 10 HAVING A NEGATIVE EXPONENT : 10^{-n} (n natural number)

1°) $\mathbf{\dfrac{1}{10\ 000} = 0.0001}$ **.**

We write : $\boxed{\mathbf{\dfrac{1}{10\ 000} = \dfrac{1}{10^4} = 10^{-4}}}$

We read : **10 power – 4** or **10 exponent – 4**.

The reciprocal of 10^4 is $\dfrac{1}{10^4} = 10^{-4}$.

2°) Observe the following chart.

$\dfrac{1}{10^n}$	**Decimal form**	**Writing in the form of a power of 10**	**Number of digits after the point**	**Exponent of 10**
$\dfrac{1}{10^2}$	0.01	10^{-2}	2	– 2
$\dfrac{1}{10^3}$	0.001	10^{-3}	3	– 3
$\dfrac{1}{10^5}$	0.00001	10^{-5}	5	– 5
$\dfrac{1}{10}$	0.1	10^{-1}	1	– 1
$\dfrac{1}{10^4}$	0.0001	10^{-4}	4	– 4

For any natural number n, we have : $\mathbf{\dfrac{1}{10^n} = 10^{-n} = \underbrace{0.0 \ldots 0}_{n \text{ zeros}}1}$

(n zeros to the left of 1).

6 PROPERTIES OF POWERS OF 10

The properties of powers of 10 are similar to those of a^n .

n and m being two natural numbers different from zero :

Properties	Examples
$\dfrac{10^n}{10^m} = 10^{n-m}$	$\dfrac{10^5}{10^3} = 10^{5-3} = 10^2$
$10^n \times 10^m = 10^{n+m}$	$10^3 \times 10^2 = 10^{3+2} = 10^5$
$(10^n)^m = 10^{n\times m}$	$(10^2)^3 = 10^{2\times 3} = 10^6$
$10^{-n} \times 10^{-m} = 10^{-(n+m)}$	$10^{-3} \times 10^{-2} = 10^{-(3+2)} = 10^{-5}$
$(10^n)^{-m} = 10^{-n.m}$	$(10^2)^{-3} = 10^{-2.3} = 10^{-6}$
$(10^{-n})^m = 10^{-n.m}$	$(10^{-2})^3 = 10^{-2.3} = 10^{-6}$
$(10^{-n})^{-m} = 10^{n.m}$	$(10^{-2})^{-3} = 10^{2.3} = 10^6$

Remarks

- All powers of 10 are positive numbers : $10^3 > 0$; $10^{-3} > 0$.
- $(-10)^n$ is positive if n is even : $(-10)^2 = 100$.
- $(-10)^n$ is negative if n is odd : $(-10)^3 = -1000$.
- $-\ 10^n$ is a negative number : $-10^2 = -100$; $-10^3 = -1000$.

7 EXPANDED FORM OF A DECIMAL NUMBER

- $365 = 3 \times 100 + 6 \times 10 + 5$
  $= 3 \times 10^2 + 6 \times 10 + 5 \times 10^0$.

«$3 \times 10^2 + 6 \times 10 + 5 \times 10^0$» is the expanded form of the number 365.

- $8635.39 = 8 \times 1000 + 6 \times 100 + 3 \times 10 + 5 + \dfrac{3}{10} + \dfrac{9}{100}$
  $= 8 \times 10^3 + 6 \times 10^2 + 3 \times 10 + 5 \times 10^0 + 3 \times 10^{-1} + 9 \times 10^{-2}$.

«$8 \times 10^3 + 6 \times 10^2 + 3 \times 10 + 5 \times 10^0 + 3 \times 10^{-1} + 9 \times 10^{-2}$» is the expanded form of the number 8635.39.

8 SCIENTIFIC NOTATION

Activity

Complete :

1°) $159.43 = 1.5943 \times 10^{\cdots}$ **2°)** $0.62 = 6.2 \times 10^{\cdots}$ **3°)** $0.009 = 9 \times 10^{\cdots}$

Definition

A **positive number in scientific notation** is in the following form :

$\boldsymbol{a \times 10^p}$ where $\boldsymbol{a}$ is a **decimal** number such that $\boldsymbol{1 \leq a < 10}$ and $\boldsymbol{p}$ **is an integer.**

EXAMPLES

The number	**Its scientific notation**
42356	$= 4.2356 \times 10^4$
0.0123	$= 1.23 \times 10^{-2}$
731.463	$= 7.31463 \times 10^2$.

Application 9

Write, in scientific notation, the following numbers : 13.42 ; 0.01 ; 15×10^{-3} .

Exercises and problems

Test your knowledge

1 Without calculating, find the sign of each of the following numbers .

$(-1.1)^3$; $(-1.2)^4$; $(-0.5)^5$; $(-20)^4$; $(-1)^{17}$; $(-1)^{10}$; $(-1)^0$;
$-(-7)^{175}$; -4^{204} ; -6^{203} .

2 Write in the form of a single power then simplify if it's possible.

$A = (-3)^2 \times (-7)^2$. $B = \left(\frac{1}{2}\right)^4 \times \left(\frac{4}{3}\right)^4$. $C = (-2)^3 \times \left(\frac{5}{2}\right)^3$.

$D = \left[\left(\frac{-3}{5}\right)^4\right]^5$. $E = (-2.1) \times (-2.1)^2 \times (-2.1)^3$. $F = \dfrac{\left(\frac{3}{5}\right)^7 \times (2.1)^7}{(-2)^7}$.

$G = \dfrac{\left(\frac{3}{5}\right)^4}{\left(\frac{-2}{5}\right)^4}$. $H = \left[\left(\frac{a}{b}\right)^2\right]^3 \times \left(\frac{a}{b}\right)$. $I = (a^3 \times a)^2$.

3 Write in the form of a power of 10.

1º) $10^2 \times 10^5$. **2º)** $10^{-1} \times 10^{-3}$. **3º)** $10^3 \times 10^{-4}$.

4º) $10^4 \times 10^{-3}$. **5º)** $10^{-1} \times 10^{-2} \times 10^{-3}$. **6º)** $10^{-4} \times 10^6 \times 10^9$.

7º) $\dfrac{10^{-2}}{10^{-5}}$. **8º)** $\dfrac{10^{-3}}{10^4}$. **9º)** $\dfrac{10^7}{10^{-4}}$.

4 Complete by a suitable exponent.

1º) $(-6)^{24} = (-6)^{13} \times (-6)^{\cdots}$. **2º)** $(1.5)^6 = \dfrac{(-1.5)^9}{(-1.5)^{\cdots}}$.

3º) $[(-4.7)^{\cdots}]^5 = -(4.7)^{15}$. **4º)** $(0.035) \times 10^{\cdots} = 35$.

5º) $\dfrac{(10)^2}{10^{\cdots}} = 10^{-8}$. **6º)** $10^3 \times 10^{\cdots} = 10^{-5}$.

7º) $[(-3.3)^4]^{\cdots} = (3.3)^{12}$. **8º)** $(-2.5) \times 10^{\cdots} = -2500$.

9º) $\dfrac{1}{10^{\cdots}} = 10^{-3}$. **10º)** $(a^3)^{\cdots} = a^9$.

5 Explain why 5^{18} is a multiple of 5^3 ; of 5^7 .

6 Explain why 7^{15} is divisible by 7^3 ; by 7^2 .

7 Write the decimal form of each of the following numbers.

1º) 10^2 . **2º)** 10^{-3} . **3º)** 10^7 . **4º)** 10^{-8} . **5º)** 10^0 .

6º) 10^{-1} . **7º)** 24×10^2 . **8º)** 1.3×10^{-3} . **9º)** 0.12×10^4 .

8 Frame each of the following numbers between two consecutive powers of 10.

301 ; 105.31 ; 0.47 ; 0.03 .

9 Write in scientific notation each of the following numbers.

0.1 ; 0.02 ; 470 ; 0.002 ; 36×10^{-3} ; $2^2 \times 3 \times 10^4$; $\dfrac{(2.5)^2 \times (4.5)^2}{(0.2)^4 \times (1.5)^4}$.

10 Write in the form of a power of 10.

1º) 2×500 . **2º)** $0.4 \times 2\,500$. **3º)** $\dfrac{1}{0.0001}$.

4º) 80×0.125 . **5º)** $\dfrac{1}{625 \times 1.6}$. **6º)** $\dfrac{1}{4} \times \dfrac{1}{25}$.

11 Calculate using a calculator.

1º) $9^4 - 8^4 - 7^4$. **2º)** $7^4 - 6^2 - 5^3$. **3º)** $5^3 - 7^2 - 8^3$. **4º)** $5^5 - \left(4^2 + 3^4\right)$.

12 Answer by true or false.

1º) If m and n are two natural numbers, then $(-3)^n \times (-3)^m = (-3)^{m+n}$.

2º) If m and n are two natural numbers, then $(-3 \times 5)^{m+n} = (-3)^m \times (5)^n$.

3º) For any natural number n, $(-5)^n$ is negative.

4º) For any natural number n, $(3-2)^n = 3^n + (-2)^n$.

5º) $(-7)^8 = 7^8$.

6º) $(-0.1)^5 = (0.1)^5$.

7º) $-5 \times (-1.3)^5 = 5 \times (1.3)^5$.

8º) $\dfrac{(-5.3)^8}{(-3.5)^8} = \left(\dfrac{5.3}{3.5}\right)^8$.

9º) $[-(-7.2)^6]^7 = (7.2)^{42}$.

10º) $(-3.4)^7 \times (2)^7 = -(6.8)^7$.

11º) $\dfrac{(-4.1)^9}{(4.1)^3} = (4.1)^6$.

12º) $10^{-5} > 10^{-4}$.

13º) $(0.5)^4 < (0.5)^3$.

13 Only one answer is correct. Which one ? Justify.

1º) $10^{-2} = ...$ **a)** 0.1 **b)** 100 **c)** 0.01

2º) $0.42 = ...$ **a)** 4.2×10^{-1} **b)** 42×10^{-1} **c)** 0.42×10^2

3º) $(-3)^5 = ...$ **a)** 3^5 **b)** -3^5 **c)** -15

4º) $(-10)^0 = ...$ **a)** -10 **b)** -1 **c)** 1 .

For seeking

14 **1°)** Write the expanded form of the number 9 876.543 using powers of 10.

2°) Find the missing digits in the following number and in its expanded form :

$3\square 56.\square 89 = 3 \times 10^3 + 4 \times 10^2 + \square \times 10^1 + 6 \times 10^0 + 7 \times 10^{-1} + 8 \times 10^{-2} + \square \times 10^{-3}$.

15 **1°)** What is the area of a rectangle R_1 having dimensions $\omega = 10^3$ cm and $L = 10^5$ cm ?

2°) A rectangle R_2 has the same area as R_1 . Its length is 10^7 cm. What is its width ?

3°) A rectangle R_3 has as dimensions 2^{10} cm and 5^{10} cm. Does it have the same area as R_1 ?

4°) A square has the same area as R_1 . What is the measure of its side ?

16 The numbers $A = 4.1 \times 10^6$, $B = 1.999 \times 10^{-4}$, $C = 1 \times 10^{-3}$ and $D = 1 \times 10^5$ are written in scientific notation.

1°) Write the decimal form of each number.

2°) Frame each number between consecutive powers of 10.

3°) Write $A \times C$ in scientific notation.

17 **1°)** What is the decimal form of $\frac{4}{5}$?

2°) What is the scientific notation of $\frac{4}{5}$?

3°) Calculate $\frac{4 \times 10^5}{5 \times 10^3}$, then write the answer in scientific notation.

18 Write the simplest form of each of the following expressions.

$$A = \frac{3^5 \times 5^3 \times 8^6}{5^2 \times 6^5 \times (10)^9} \quad ; \quad C = \left(\frac{-3}{5}\right)^7 \times \left(\frac{-8}{3}\right)^5 \times \left(\frac{0.2}{0.5}\right)^6 .$$

$$B = \frac{(-7)^9 \times (-12)^9 \times 2^6}{(21)^7 \times 6^5} \quad ; \quad D = \frac{(7.2)^3 \times (3.6)^2}{24.3 \times (4.8)^4 \times 72^2} .$$

19 Given the numbers :

$$A = \left[\left(\frac{-2}{3}\right)^5 \times \left(\frac{-2}{3}\right)^3\right] \div \left(\frac{2}{3}\right)^6 . \qquad C = \left[\left(\frac{-3}{4}\right)^2 \times \left(\frac{2}{3}\right)^3\right]^2 .$$

$$B = \left[\left(\frac{-3}{5}\right)^8 \div \left(\frac{-3}{5}\right)^6\right] \times \left(\frac{3}{5}\right)^4 . \qquad D = \left[\left(\frac{-5}{3}\right)^4 \div \left(\frac{5}{3}\right)^3\right]^2 .$$

1º) Precise the sign of A, B, C and D.

2º) Develop and simplify each one of the given numbers.

20 Simplify.

$$A = \frac{10^2 \times 18^3}{15^3 \times 12^2} \quad ; \quad B = \frac{(-6)^3 \times 10^2}{(-8)^3 \times 10^{-1} \times 15^2} \quad ; \quad C = \frac{4.5 \times 0.21 \times 10^3}{0.5 \times 70}$$

$$D = \frac{\left(\frac{2}{3}\right)^3 \times \left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)^3 \times \left(\frac{2}{3}\right)^4} \; ; \; E = \frac{(2.25)^3 \times (4.9)^4 \times (0.8)}{(0.14)^3 \times (10.5)^5} \; ; \; F = \frac{(-0.2)^4 \times (0.5)^3 \times (0.8)^2 \times 10^5}{(0.04)^2 \times (-2)^3 \times (-5)^2}$$

TEST

1 Calculate and reduce the following expressions.

$\mathbf{A} = (-3.4)^2 \times (-2.5)^2$.

$\mathbf{D} = \dfrac{(5)^2 \times (-2)^3 \times 10^2}{(-4)^3 \times 10^{-2}}$.

$\mathbf{B} = (-3.4)^4 \div (3.4)^2$.

$\mathbf{E} = (-4) \times (-0.5)^2 \times 2 \times (0.5)$.

$\mathbf{C} = [(-0.2)^3]^9$.

(2 - 2 - 2 - 3 - 3 points)

2 Calculate (write the answer in decimal form).

$$\frac{(-0.2)^4 \times (0.5)^3 \times (0.8)^2 \times 10^5}{4^2 \times (-2) \times (-5)^2} .$$

(2 points)

3 Write in scientific notation.

$0.0413 \quad ; \quad \dfrac{1}{25} \quad ; \quad \dfrac{2}{5} \quad ; \quad 1\,999.001.$

(4 points)

4 Verify that each of the following numbers is written in the form $a^n \times b^m \times 10^p$, where n, m and p are natural numbers. **(2 points)**

$\mathbf{A} = (0.2)^4 \times (-0.5)^3 \times (-2) \times (-0.2)^3 \times 5^3 \times 10^{-4}$.

$\mathbf{B} = \dfrac{(-6)^3 \times (8)^4 \times (0.3)^4}{(0.3)^3 \times 9^2 \times (0.2)^2}$.

5 The planet Uranus is 29×10^8 km far from the sun and has a diameter of 51×10^3 km.

Write these numbers in the form of natural numbers. **(2 points)**

2

GREATEST COMMON FACTOR AND LEAST COMMON MULTIPLE OF TWO OR MORE NATURAL NUMBERS

Objective

Calculate the GCF and the LCM of two or more natural numbers.

CHAPTER PLAN

COURSE

1. Relation between the *GCF* and the *LCM* of two natural numbers
2. Finding the *GCF* of three or more natural numbers
3. Finding the *LCM* of two natural numbers (reminder)

EXERCISES AND PROBLEMS

TEST

1 RELATION BETWEEN THE *GCF* AND THE *LCM* OF TWO NATURAL NUMBERS

Activity

1º) Determine d, the greatest common factor (GCF), and m, the least common multiple (LCM), of these two numbers : 108 and 405.

2º) Verify that : $m \times d = 108 \times 405$.

Ruler

If d is the GCF of two natural numbers a and b, and m is their LCM , then :

$$\boxed{m \times d = a \times b}$$

In particular, if a and b are relatively prime numbers then $d = 1$ and $m = a \times b$.

2 FINDING THE *LCM* OF THREE OR MORE NATURAL NUMBERS

Activity

1º) Write each of the following numbers as a product of prime factors : 390 , 630 and 825.

2º) Determine $d_1 = GCF\ (390\ ,\ 630)$; write d_1 in the form of a product of prime factors.

3º) Determine d, the GCF of d_1 and 825; write d in the form of a product of prime factors.

4º) Determine the product e of the common prime factors of the numbers 390 , 630 and 825 .

5º) Do you obtain $d = e$?

Ruler

To find the GCF of three or more natural numbers, use one of the following methods :

1st method Calculate d_1 , the *GCF* of two of these numbers, then the *GCF* of d_1 with the third number, and so on ...

2nd method

⊙ Write the prime factorization of each number.

⊙ Calculate the product of the common prime factors.

Remark

If the *GCF* of two or more numbers is 1, then they are called relatively prime numbers (these numbers are not necessarily prime).

The three numbers 322 , 546 and 1 045 are relatively prime because of the following :

$322 = 2 \times 7 \times 23$; $546 = 2 \times 3 \times 7 \times 13$; $1\,045 = 5 \times 11 \times 19$.

GCF (322 , 546 and 1 045) = 1 .

Application 1

1°) Find $d = GCF$ (126 , 230 , 420).

2°) Verify that the three numbers 78 , 120 and 85 are relatively prime numbers.

3°) Calculate the *GCF* of the following numbers : 48 , 108 , 150 and 225 .

3 FINDING THE *LCM* OF THREE OR MORE NATURAL NUMBERS (reminder)

Activity

1°) Write each of the following numbers as a product of prime factors : 168 , 180 and 252.

2°) Determine $m_1 = LCM$ (168 , 180) ; write m_1 in the form of a product of prime factors.

3°) Determine m, the *LCM* of m_1 and 252 ; write m in the form of a product of prime factors.

4°) Determine the product f of all the common prime factors of the numbers 168 , 180 and 252 , and the remaining factors.

5°) Would you obtain $m = f$?

Ruler

To find the *LCM* of three or more natural numbers, use one of the following methods :

1st method Calculate m_1, the *LCM* of two of these numbers, then the *LCM* of m_1 with a third number, and so on ...

2nd method
- ⊙ Write the prime factorization of each number.
- ⊙ Calculate the product of the highest power of each prime factor.

Application 2

1º) Find the *LCM* of the numbers 840 , 308 and 675.

2º) Calculate the *GCF* of 112 and 135 ; what is their *LCM* then ?

EXERCISES AND PROBLEMS

Test your knowledge

1 a and b are two natural numbers having d as *GCF* and m as *LCM* .
Complete the following chart :

a	b	d	m	$a \times b$	$d \times m$
24	30				
16	48				
19	36				
25	26				

2 **1º)** Calculate the *GCF* and the *LCM* of these two numbers :

$a = 2^3 \times 3 \times 7$ and $b = 5 \times 17$.

2º) Compare the obtained *LCM* and the product $a \times b$.

3 Find the *GCF* of 25 and 39 ; deduce their *LCM* .

4 Calculate the *GCF* of the numerator and the denominator of the following fractions, then write each in an irreducible form.

1º) $\dfrac{84}{96}$ **3º)** $\dfrac{126}{588}$

2º) $\dfrac{99}{77}$ **4º)** $\dfrac{2\,500}{3\,600}$.

5 **1º)** Determine d and m, the *GCF* and the *LCM* of 852 and 1 314 .

2º) Find x and y knowing that $852 = d \times x$ and $1\,314 = d \times y$.

3º) Verify that x and y are relatively prime numbers.

6 **1º)** $d = GCF\ (a\ ;\ b)$, $m = LCM\ (a\ ;\ b)$. Complete the following table.

a	*b*	*d*	*m*
5	6		
16	17		
20	21		
3	4		

2º) n and $n + 1$ are two consecutive natural numbers ($n \neq 0$). Can you guess their *GCF* ? their *LCM* ?

7 **1º)** Calculate the *GCF* of 48 and 64 ; let it be d.

2º) Calculate the *GCF* of d and 72 ; let it be ℓ .

3º) What does ℓ represent for the numbers 48 , 64 and 72 ?

4º) Follow the same method to calculate the *GCF* of 135 , 25 and 75.

8 Write as a product of prime factors then calculate the *GCF* and the *LCM* of the following :

1º) 3 600 ; 4 050 and 540 .

2º) 81×35 ; 63×7 and 72×25 .

3º) $8 \times 9 \times 13$; $15 \times 56 \times 9$ and $27 \times 169 \times 343$.

9 **1º)** Verify that 24 ; 36 and 17 are relatively prime numbers.

2º) Calculate the *LCM* of 24 ; 36 and 17.

10 Find the *LCM* of the following :

1º) 15 ; 20 and 60 .

2º) 8 ; 18 and 63 .

11 **1º)** Calculate $d = GCF\ (60\ ,\ 80\ ,\ 108)$ and $m = LCM\ (60\ ;\ 80\ ;\ 108)$.

2º) Find D the GCF and M the LCM of the following :
7×60 ; 7×80 and $M = 7 \times m$.

3º) Verify that $D = 7 \times d$ and $M = 7 \times m$.

12 Answer by True or False.

1º) The GCF of many numbers divides each of these numbers.

2º) The GCF of two natural numbers is greater than each of these numbers.

3º) Two natural numbers, which are relatively prime, are prime numbers.

4º) Two prime numbers are always relatively prime.

5º) The two following numbers :
$a = 2^3 \times 3^2 \times 5^2$ and
$b = 5 \times 7 \times 11 \times 13^2$ have 7 as GCF .

6º) The LCM of two natural numbers a and b is less than each one of them.

7º) 12 is a multiple of 4, therefore :
$LCM\ (12\ ,\ 4) = 12$.

8º) 4 is a divisor of 12, therefore :
$GCF\ (12\ ,\ 4) = 4$.

For seeking

13 The GCF of two natural numbers is 154 ; their LCM is 4 620 .
If one of them is 770 , find the other one.

14 Find two relatively prime numbers knowing that their sum is 8 . Give all the possible solutions.

15 The product of two natural numbers is 1 512 ; their LCM is 252 .
Find their GCF .

16 **1º)** Write each of these numbers as a product of prime factors : 506 , 759 and 177 .

2º) Calculate the LCM of the above numbers; let it be m .

3º) Find the quotients of the division of m by each of these numbers : 506 , 759 and 177.

4º) Calculate the GCF of the obtained quotients.

5º) Are these quotients relatively prime numbers ?

17 Three pieces of land have respective areas : 3 000 m^2 , 3 600 m^2 and 4 800 m^2 . The three pieces are sold according to these conditions :

1º) The pieces are divided into lots all having the same area.

2º) The area of each lot is the greatest possible.
Calculate the number of lots for each piece of land.

18 Three ships leave the same port : The first every 6 days, the second every 8 days and the third every 12 days.

If they leave together on the 5^{th} of May, on which day would they leave again together ?

19 A lighthouse emits three different lights : a red light every 12 seconds, a green light every 15 seconds, and a yellow light every 18 seconds. These lights are emitted simultaneously.
After how many seconds would the three lights be emitted together again ?

20 A salesman has three pieces of fabric : the first is 315 cm, the second is 525 cm and the third is 630 cm.
He wants to divide them into the longest possible equal pieces. What would be the common length of these pieces ?

21 For a school festival, the students are arranged in exact lines of 5, of 6, and of 10.
Find the number of the students knowing that the number is between 400 and 500 .

22 Apple trees, equally spaced, were planted all around a triangular piece of land with the sides measuring 144 m, 180 m and 240 m.
Knowing that there is a tree at each top and that the distance between two consecutive trees is between 4 m and 10 m, calculate the number of planted apple trees.

23 The number of stamps that Nabil has is between 970 and 1 000.

If he arranges them by groups of 5, 6 or 10, 4 always remain.

What is the number of these stamps ?

24 A group of soldiers that are less than 1000 men, are exercising in a field .

While seeing them standing in columns :
1°) per line of 8 men,
2°) per line of 15 men,
3°) per line of 25 men,

a spectator has noticed that in the three cases the last line was incomplete and with only 5 men. This spectator affirms that if the soldiers stand in columns per line of 11 men, all the lines will be complete.

Find the number of the soldiers.

25 By counting the steps of a staircase 2 by 2, 1 remains. By counting them 3 by 3, 2 remain; by counting them 5 by 5, 4 remain.
Find the number of the steps of this staircase, knowing that it is between 70 and 100.

26 Marc counts his stamps by 12, by 16 and by 20; at each time, 8 remain. By counting them by 13, nothing remains.

How many stamps does Marc have ?

TEST

1 *a* and *b* are two natural numbers; *d* and *m* are their *GCF* and *LCM* respectively.

1º) Complete the chart. **(2 points)**

a	*b*	*d*	*m*
7	9		
9	11		
21	23		
23	25		

2º) Complete. **(1 point)**

If *a* and *b* are two consecutive odd numbers, then :

GCF (*a* , *b*) = ... and *LCM* (*a* , *b*) = ...

2 **1º)** Calculate $x = GCF\ (420 , 3\,080)$.

2º) Calculate $y = GCF\ (x , 455)$.

3º) What does *y* represent for the following numbers : 420 , 3 080 and 455 ? **(3 points)**

3 **1º)** Write each of these numbers as a product of prime factors :
345 , 880 and 1 260 .

2º) Find *d* the *GCF* , and *m* the *LCM* of these numbers.

3º) Find the quotients, by *d*, of each of these numbers : 345 , 880 and 1 260 .

4º) Verify that these quotients are relatively prime numbers. **(5 points)**

4 The *GCF* of two numbers is 28, their *LCM* is 27 720 ; One of them is 2 520 .
Find the other one. **(2 points)**

5 Write the fraction $\frac{825}{930}$ in the form of an irreducible fraction. **(2 points)**

6 **1º)** Write each of these numbers as a product of prime factors :
540 , 420 and 1 170.

2º) Calculate *d* the *GCF* , and *m* the *LCM* of the above numbers.

3º) Find the quotients of *m* by each of these numbers : 540 , 420 and 1 170.

4º) Verify that the obtained quotients are relatively prime numbers. **(5 points)**

3

CONGRUENT RIGHT TRIANGLES

Objective

To know and to use the cases of congruency of two right triangles.

CHAPTER PLAN

COURSE

1. 1^{st} case of congruency of two right triangles

2. 2^{nd} case of congruency of two right triangles

EXERCISES AND PROBLEMS

TEST

1 FIRST CASE OF CONGRUENCY OF TWO RIGHT TRIANGLES

Activity

1º) ⊙ Draw a segment $[BC]$ measuring 5 cm.

⊙ Draw the semi-line $[Bx)$ that forms with $[BC)$ an angle of $40°$.

⊙ Let A be the foot of the perpendicular drawn from C to $[Bx)$.

Thus, you have constructed a right triangle *ABC* knowing the measure of its hypotenuse and an acute angle.

2º) Do the same to construct a triangle MNF right at M and such that

$NF = 5$ cm and $\widehat{FNM} = 40°$.

3º) Make an exact copy of each of the two triangles drawn above.

4º) Verify that the two copies are congruent.

5º) Name the congruent sides of these two triangles.

Rule

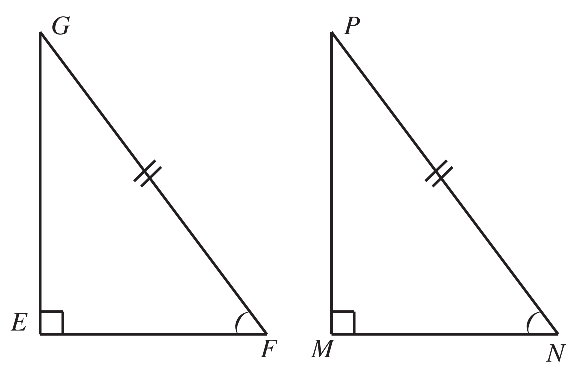
If **the hypotenuse** and an acute angle of one **right triangle** are **equal** to the hypotenuse and an **acute angle** of a second right triangle, then the two triangles are **congruent**.

EXAMPLE

The two triangles EFG and MNP have :

$FG = NP$ and $\widehat{GFE} = \widehat{PNM}$.

Therefore they are congruent.



Application 1

(xy) is a line passing through the midpoint O of a segment $[AB]$. C and D are the orthogonal projections of A and B on (xy) respectively.

Prove that the two triangles AOC and BOD are congruent.

2 SECOND CASE OF CONGRUENCY OF TWO RIGHT TRIANGLES

Activity

1º) Draw a right angle $\widehat{xAy}$. On one of its sides, [Ax) for example, place a point B such that AB = 2.5 cm, and on the second side, [Ay) , place a point C such that BC = 4 cm.

Therefore, you have constructed a right triangle *ABC* knowing the measure of its hypotenuse and a side of the right angle.

2º) Do the same to construct a triangle KJL right at K and such that :
KJ = 2.5 cm and JL = 4 cm.

3º) Make an exact copy of each of the two triangles drawn above.

4º) Verify that the two copies are congruent.

5º) Name the equal angles of both triangles.

Rule

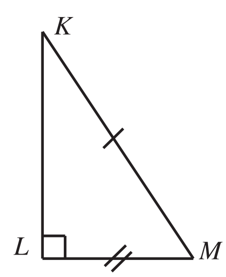
Two **right triangles**, having **the hypotenuse** and a leg of **one equal** to the hypotenuse and a leg of the second, are **congruent.**

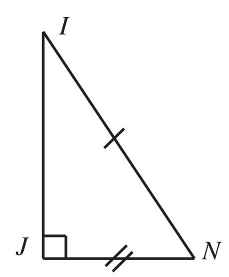
EXAMPLE

The two triangles KLM and IJN have :

KM = IN and LM= JN .

Therefore they are congruent.





Application 2

B and C are two points placed respectively on the sides [Ax) and [Ay) of an angle $\widehat{xAy}$ with AB = AC.

The perpendiculars drawn from B and C to [Ax) and [Ay) respectively, intersect at I.

Prove that the two triangles ABI and ACI are congruent.

Remark

The three cases of the congruency of any triangles can be applied to right triangles.

In particular : two right triangles are congruent when the legs of one triangle are respectively congruent to the legs of the other.

Application 3

Let $[Or)$ be the bisector of angle $\widehat{xOy}$. E is a point on $[Or)$.

The perpendicular drawn from E to $[Or)$ cuts $[Ox)$ and $[Oy)$ at H and K respectively.

Prove that the two triangles OEH and OEK are congruent.

EXERCISES AND PROBLEMS

Test your knowledge

1 Construct a right isosceles triangle SOL of vertex O such that $OS = 42$ mm.

2 Construct a triangle LAC right at A such that :

$LA = 6$ cm and $AC = 8$ cm.

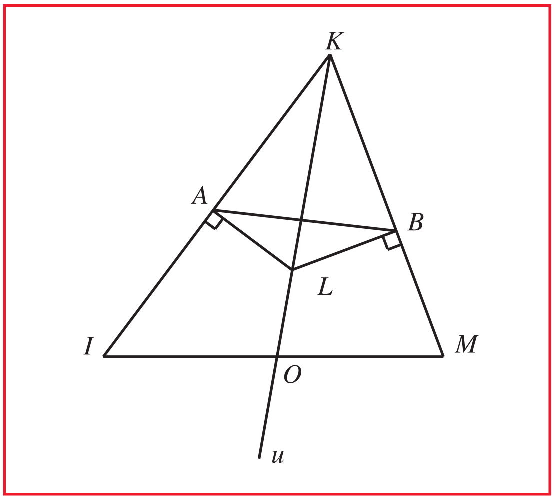
3 Construct a triangle SUD right at D such that :

$SU = 4$ cm and $\widehat{USD} = 48°$.

4 In triangle KIM, the bisector $[Ku)$ cuts $[IM]$ at O. From a point L on $[KO]$, draw the perpendiculars (LA) and (LB) to (KI) and (KM).

1°) Show that : $KA = KB$.

2°) Show that : $(KL) \perp (AB)$.



5 BEL is an isosceles triangle of vertex B. $[ET]$ and $[LS]$ are the altitudes relative to the equal sides.

1°) Prove that $ET = LS$.

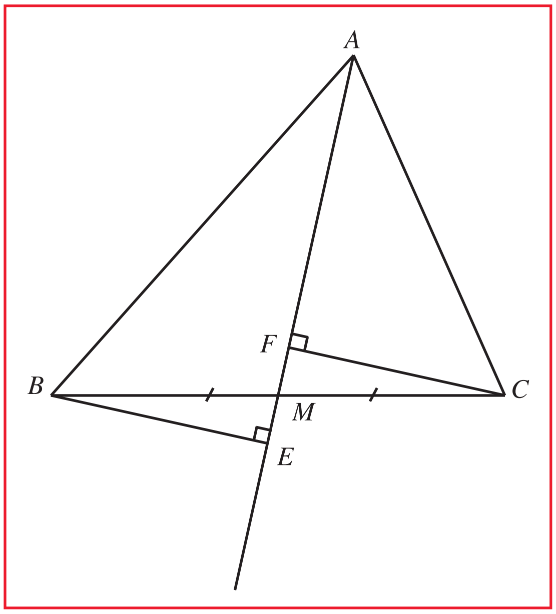
2°) Deduce that $ES = LT$ and that triangle BST is isosceles.

6 $[AM]$ is the median relative to $[BC]$ in a triangle ABC. E and F are the orthogonal projections of B and C on line (AM).

1°) Prove that $BE = CF$ and that M is the midpoint of $[EF]$.

2°) Prove that $CE = BF$.

3°) Prove that the two triangles BEF and CEF are congruent.



7 On the sides $[Ax)$ and $[Ay)$ of an angle $\widehat{xAy}$, place the points B and C respectively such that $AB = AC$. L and S are the feet of the perpendiculars drawn from B and C to $[Ay)$ and $[Ax)$ respectively.

1º) Prove that $BL = SC$ and that $AS = AL$.
Deduce that $BS = LC$.

2º) $[BL]$ and $[CS]$ intersect at O. Prove that the two triangles BOS and COL are congruent. Deduce that O is a point of the bisector of $\widehat{BAC}$.

3º) Prove that (AO) is the perpendicular bisector of $[BC]$ and of $[SL]$.

8 Answer by true or false.

1º) In a right triangle, the two sides are perpendicular.

2º) In a right triangle, each side is called hypotenuse.

3º) Two right triangles are congruent when the hypotenuse of one is congruent to the hypotenuse of the other.

4º) Two right triangles are congruent when the acute angles in one are respectively equal to the acute angles in the other.

5º) In a right triangle, the acute angles are complementary.

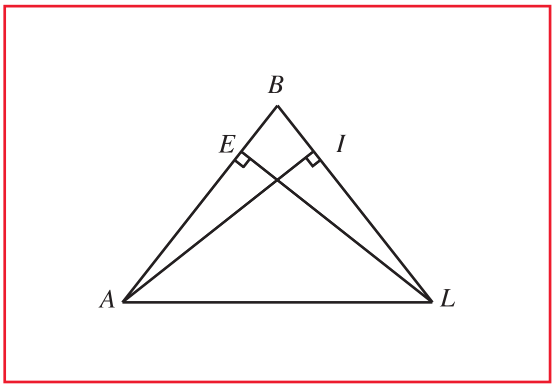
For seeking

9 **1º)** Construct a triangle BON such that :
$BN = 5$ cm, $BO = 3$ cm and $NO = 4$ cm.

2º) Measure angle $\widehat{BON}$.

10 In a triangle BAL, the altitudes $[AI]$ and $[LE]$ are congruent.

Prove that triangle BAL is isosceles.



11 On the sides $[Ex)$ and $[Ey)$ of an angle $\widehat{xEy}$, we place the points B and S respectively such that $EB = ES$. The perpendicular from B to $[Ex)$ cuts the perpendicular from S to $[Ey)$ at L.

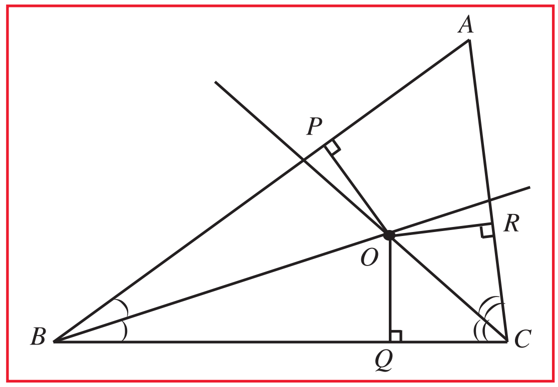
1º) Prove that $LB = LS$.

2º) Which fixed line does point L describe when B and S vary on $[Ex)$ and $[Ey)$ while being equidistant from E ?

12 ABC is any triangle.

The bisectors of angles $\widehat{ABC}$ and $\widehat{ACB}$ intersect at O.

P, Q and R are the feet of the perpendiculars drawn from O to the lines (AB), (BC) and (AC) respectively.



1º) Prove that O is equidistant from (AB) and (AC).

2º) Deduce that the bisectors of the angles of the triangle intersect at the same point.

3º) Prove that the circle of center O and radius OP passes through R and Q.

13 $\widehat{xTy}$ is an acute angle; O is a point on $[Tx)$ and I a point on $[Ty)$.

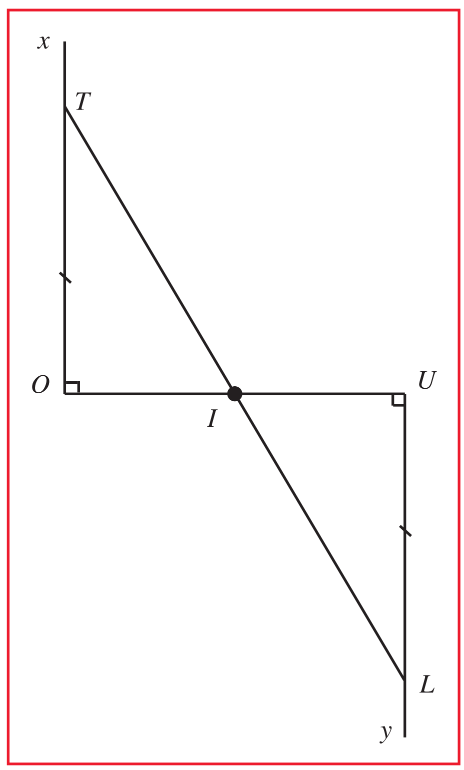
1º) Construct the bisectors $[Os)$ and $[Iv)$ of angles $\widehat{xOI}$ and $\widehat{yIO}$. $[Os)$ and $[Iv)$ intersect at R.

2º) Draw the perpendiculars $[RA]$, $[RE]$ and $[RD]$ to (Tx), (OI) and (Ty) respectively.
Prove that $RA = RE = RD$

14 $[OU]$ is a segment. On both sides of $[OU]$, draw the semi-lines $[Ox)$ and $[Uy)$ perpendicular to (OU).

Place a point T on $[Ox)$ and a point L on $[Uy]$ such that $OT = UL$.

If I is the midpoint of $[OU]$, show that the two triangles TOI and LUI are congruent and deduce that the points T, I and L are collinear.



TEST

1 ABC is an isosceles triangle of vertex A. H is the midpoint of $[BC]$. M is a point on line (BC) such that : $CM = CA$ with C being between M and B. E is the foot of the perpendicular drawn from M to (AC).

1º) Prove that $ME = AH$.

(4 points)

2º) Let F be the symmetric of C with respect to E. Show that the two triangles ABC and MCF are congruent.

(4 points)

2 ABC is a right triangle at A with $AB < AC$. $[Bx)$ is the semi-line drawn inside the triangle such that $\widehat{CBx} = \widehat{ACB}$. A' is the foot of the perpendicular drawn from C to $[Bx)$.

1º) Prove that the two triangles ABC and $A'BC$ are congruent.

(4 points)

2º) (BA') and (AC) intersect at I. Prove that the two triangles IAB and $IA'C$ are congruent.

(4 points)

3º) (BA) and (CA') intersect at S. Prove that (SI) and (BC) are perpendicular.

(4 points)

4

SQUARE ROOTS

Objectives

1. To recognize the square roots of a positive number.

2. To find the square roots of a perfect square.

CHAPTER PLAN

COURSE

1. Reminder

2. Activity

3. Activity

4. Usage of the calculator

5. Reducing expressions containing radicals

EXERCISES AND PROBLEMS

TEST

1 REMINDER

⊙ The square of a number is always positive. $(2)^2 = 4$; $(-5)^2 = 25$.

⊙ Two opposite numbers have the same square. $(-2)^2 = 4$ et $(+2)^2 = 4$.

2 ACTIVITY

What is the positive number that has for a square the following : 4 ? 9 ? $\frac{16}{25}$? 144 ? 0.36 ? 1 ?

Definition

⊙ ***a*** being a **positive** number, there is a **positive number whose square is *a***. This number is **called the square root of *a*** and is written $\sqrt{a}$.

$$\boxed{(\sqrt{a})^2 = a} \qquad (a \geq 0)$$

The symbol $\sqrt{\ }$ is called **radical** sign.
a is the **radicand**.

⊙ ***a*** being a **positive number whose square is a^2** , $\sqrt{a^2}$ is ***a*** .

$$\boxed{\sqrt{a^2} = a} \qquad (a \geq 0)$$

EXAMPLES

$\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$; $(\sqrt{0.1})^2 = 0.1$; $\left(\sqrt{\frac{1}{2}}\right)^2 = \frac{1}{2}$; $\sqrt{5^2} = 5$;

$\sqrt{(1.2)^2} = 1.2$; $\sqrt{\left(\frac{3}{4}\right)^2} = \frac{3}{4}$; $\sqrt{25} = 5$ because $5^2 = 25$;

$\sqrt{1.69} = 1.3$ because $(1.3)^2 = 1.69$.

3 ACTIVITY

1°) What is the negative number that has for a square the following : 4 ? 9 ? $\frac{16}{25}$? 144 ? 0.36 ? 1 ?

2°) What are the numbers that have for squares the following : 4 ? 9 ? $\frac{16}{25}$? 1 ? 0.81 ? 100 ? 0 ? $\frac{49}{81}$?

Properties

⊙ $(-\sqrt{a})^2 = (+\sqrt{a})^2 = a$ (a positive).

$-\sqrt{a}$ is the **negative square root of a .**

⊙ 0 admits only one square root which is 0.

⊙ $\sqrt{1} = 1$ and $-\sqrt{1} = -1$.

A negative number does not have a square root.

EXAMPLES

⊙ 49 admits two square roots :
positive $\sqrt{49} = 7$ and negative $-\sqrt{49} = -7$.

⊙ $\sqrt{(-4)^2} = \sqrt{(4)^2} = 4$.

⊙ – 16 does not have a square root : $\cancel{\sqrt{-16}}$.

Application 1

1°) Calculate : $\sqrt{36}$; $-\sqrt{(1.2)^2}$; $\sqrt{\frac{25}{36}}$; $\left(\sqrt{29}\right)^2$; $\sqrt{a^4}$; $\sqrt{(-3)^2}$; $\sqrt{(-5)^2}$; $\sqrt{5} \times \sqrt{5}$.

2°) Complete the following charts.

a	1	3	6	15	2.5	1.7	$\frac{8}{13}$	9	0.8	1.02
$(\sqrt{a})^2$										
$\sqrt{a^2}$										

a	1	4	0	...	16	...	6^2	...	100	...
$\sqrt{a}$	1	2	0	3	...	$\sqrt{8}$	...	8	...	1.5

4 USAGE OF THE CALCULATOR

A **calculator** helps to calculate the **positive square root** of any **positive number** by using the key $\boxed{\sqrt{}}$.

For certain numbers, the calculator shows the exact value of the square root; for others, it shows their approximate value.

EXAMPLES

⊙ $\boxed{1}$ $\boxed{9}$ $\boxed{6}$ $\boxed{\sqrt{\ }}$ 14 or $\boxed{\sqrt{\ }}$ $\boxed{1}$ $\boxed{9}$ $\boxed{6}$ $\boxed{=}$ 14

(depending on the type of the calculator).

⊙ $\boxed{1}$ $\boxed{9}$ $\boxed{\sqrt{\ }}$ 4.35889844 or $\boxed{\sqrt{\ }}$ $\boxed{1}$ $\boxed{9}$ $\boxed{=}$ 4.35889844

(depending on the type of the calculator).

Without using the calculator, we notice that the last digit of the number $(4.35889844)^2$ is 6; thus we deduce that the number is different from 19.

The number 4.35889844 is then an approximate value of $\sqrt{19}$.

Therefore, we get :

$\sqrt{19} \simeq 4.3$ is to the nearest 10^{-1} ,

$\sqrt{19} \simeq 4.35$ is to the nearest 10^{-2} ,

$\sqrt{19} \simeq 4.358$ is to the nearest 10^{-3} ,

$\sqrt{19} \simeq 4.359$ is to the nearest 10^{-3} .

Application 2

1º) Use your calculator to give the exact value of the following :

$\sqrt{9}$; $\sqrt{16}$; $\sqrt{121}$; $\sqrt{2.56}$; $\sqrt{15\,129}$.

2º) Use your calculator to give the approximate value to the nearest 10^{-3} of the following : $\sqrt{2}$; $\sqrt{3}$; $\sqrt{5}$; $\sqrt{7}$; $\sqrt{13}$; $\sqrt{17}$.

5 OPERATIONS WITH RADICALS

$a \times \sqrt{b}$ **is written** $a\sqrt{b}$ **with** $b \geqslant 0$

EXAMPLES

1°) $7\sqrt{2} - 3\sqrt{2} = (7 - 3)\sqrt{2} = 4\sqrt{2}$.

2°) $11\sqrt{13} - 20\sqrt{13} + 9\sqrt{13} = (11 - 20 + 9)\sqrt{13} = 0 \times \sqrt{13} = 0$.

3°) $3\sqrt{5} - 2\sqrt{3} + 7\sqrt{5} - 10\sqrt{3} = (3+7)\sqrt{5} + (-2-10)\sqrt{3} = 10\sqrt{5} - 12\sqrt{3}$.

4°) $\sqrt{a} + \sqrt{a} = 2\sqrt{a} \quad (a \geq 0)$.

5°) $3\sqrt{a} + 4\sqrt{b} - 2\sqrt{a} - 9\sqrt{b} = (3-2)\sqrt{a} + (4-9)\sqrt{b} = \sqrt{a} - 5\sqrt{b}$
$(a \geq 0$ and $b \geq 0)$.

6°) $\left(2 - \sqrt{3}\right)\left(2 + \sqrt{3}\right) = 2^2 + 2\sqrt{3} - 2\sqrt{3} - \left(\sqrt{3}\right)^2 = 4 - 3 = 1$.

7°) $\left(\sqrt{11} - 5\right)\left(\sqrt{11} + 5\right) = \left(\sqrt{11}\right)^2 + 5\sqrt{11} - 5\sqrt{11} - 25 = 11 - 25 = -14$.

Application 3

Reduce $(x \geqslant 0,\ a \geqslant 0$ and $b \geqslant 0)$.

1°) $5\sqrt{6} - 19\sqrt{6} + \sqrt{6}$.

2°) $2\sqrt{3} - 5\sqrt{2} + 9\sqrt{3} + 7\sqrt{2}$.

3°) $4\sqrt{x} - 3\sqrt{x} + 21\sqrt{x} - 10\sqrt{x}$.

4°) $-13\sqrt{a} + 4\sqrt{b} + 3\sqrt{a} + 9\sqrt{b}$.

5°) $\left(\sqrt{7} - 3\right)\left(\sqrt{7} + 3\right)$.

6°) $3\left(1 - \sqrt{5}\right) + 2\sqrt{5}\left(1 + \sqrt{5}\right)$.

EXERCISES AND PROBLEMS

Test your knowledge

1 **1°)** What is the square of $\sqrt{34}$?

2°) What is the value of $\left(\sqrt{34}\right)^2$?

2 **1°)** What are the numbers whose square is 9 ? 3 ?

2°) Is there a number whose square is – 3 ?

3 Reduce the following radicals.

$\sqrt{9}$; $\sqrt{\dfrac{4}{25}}$; $\sqrt{10\,000}$; $\sqrt{0.81}$; $\left(\sqrt{15}\right)^2$; $\dfrac{\sqrt{1}}{\sqrt{4}}$; $\sqrt{(-10)^2}$.

4 Calculate.

$\sqrt{(-15)^2}$; $\sqrt{0}$; $\sqrt{+81}$; $\sqrt{(-7)^2}$; $\left(-\sqrt{36}\right)^2$; $-\sqrt{(-0.1)^2}$.

5 Simplify.

$\left(\sqrt{3}\right)^2$; $\left(-\sqrt{3}\right)^2$; $-\sqrt{9}$; $\sqrt{(-5)^2}$; $-\sqrt{(5)^2}$.

6 Complete the following charts.

x	– 7	– 5	– 2	0	+ 2	+ 5	+ 7
x^2							

x	49		100		$(23)^2$		$(-8)^2$
$\sqrt{x}$		5		$\sqrt{7}$		13	

7 Complete the following chart.

a	-3	-1	0.2	-7	$\frac{-2}{5}$	$+7$	$\frac{+2}{5}$	$+\sqrt{3}$	$-\sqrt{3}$	$+3$
a^2										
$\sqrt{a^2}$										

8 Complete.

1º) ... is the square of 3 .

2º) 16 is the square of ... .

3º) 16 has for square ... .

4º) ... is the square root of 25 .

5º) ... is the positive square root of 11 .

9 Complete the following chart.

a	7	49	0	19	1.44
Positive square root of a					
Negative square root of a					
Square roots of a					

10 Use your calculator to give the approximate value of the following :

1º) $\sqrt{5}$ to nearest 10^{-2} .

2º) $\sqrt{3}$ to nearest 10^{-3} .

3º) $\sqrt{2}$ to nearest 10^{-2} .

4º) $\sqrt{7}$ to nearest 10^{-3} .

5º) $\sqrt{15}$ to nearest 10^{-1} .

6º) $\sqrt{17}$ to nearest 10^{-2} .

11 Write the following expressions in the form of $a\sqrt{b}$ $(b \geq 0)$.

$A = 5\sqrt{2} - 3\sqrt{2}$.

$B = 8\sqrt{5} - 17\sqrt{5}$.

$C = 3\sqrt{7} - 4\sqrt{7} - 8\sqrt{7} + 5\sqrt{7}$.

$D = 3\sqrt{x} + 2\sqrt{x} - \sqrt{x}$ $(x \geq 0)$.

$E = 5\sqrt{y} - 12\sqrt{y} + 17\sqrt{y} - 10\sqrt{y}$ $(y \geq 0)$.

12 Find the positive number a.

1º) $\sqrt{a} = 11$. **2º)** $\sqrt{a} = 9$. **3º)** $\sqrt{a} = 0$. **4º)** $\sqrt{a} = \frac{9}{4}$.

13 Circle the correct answer in each case.

$\sqrt{25} =$	-5	12.5	5
$\sqrt{-4} =$	-2	wrong form	2
3 is the square root of	$\sqrt{9}$	-9	9
$\frac{-3}{2} =$	$\sqrt{\frac{9}{4}}$	$\frac{6}{4}$	$\frac{\sqrt{9}}{-\sqrt{4}}$
$\sqrt{256} =$	128	16	-16
$\sqrt{10\,000} =$	100	-100	$5\,000$

For seeking

14 Calculate and reduce.

1º) $\sqrt{3}(\sqrt{3} + 1)$.

2º) $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$.

3º) $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$.

4º) $(3\sqrt{2})^2 - 2\sqrt{2}$.

5º) $(\sqrt{2} - y)(\sqrt{2} + y)$.

6º) $(11 + \sqrt{15})(11 - \sqrt{15})$.

7º) $(4 - \sqrt{x})(4 + \sqrt{x})$ $(x \geq 0)$.

8º) $(3\sqrt{2})^2 - (2\sqrt{2})^2$.

15 Let $a = \sqrt{19 - x}$.

1º) Calculate a if $x = 3$.

2º) Can you find the numerical value of a if $x = 28$?

3º) Find the value of x when $a = 5$.

16 Consider $E = 2x^2 - \sqrt{3}\,x + 4\sqrt{3}$.

Calculate the numerical value of E when :

1º) $x = 0$. **2º)** $x = 1$. **3º)** $x = \sqrt{3}$.

17 Write in the form of a power.

$\sqrt{10^4}$; $\sqrt{10^6}$; $\sqrt{10^8}$; $\sqrt{10^{10}}$.

18 Calculate $A = \sqrt{10^2} + \sqrt{10^4} + \sqrt{10^6}$.

19 Calculate $A = \sqrt{x^2 + y^2}$ and $B = \sqrt{x^2} + \sqrt{y^2}$, in each of the following cases.

1º) $x = 4$ and $y = 3$.

2º) $x = 4$ and $y = -3$.

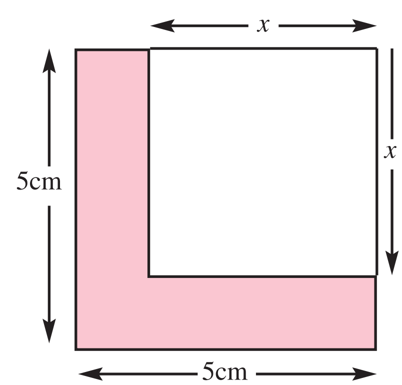
20 Give a number equal to its positive square root.

21 Given a triangle ABC such that :

$AB = \sqrt{5}$, $BC = \sqrt{20} - \sqrt{5}$ and $CA = \sqrt{45} - \sqrt{20}$.

Is it equilateral ? Justify.

22 In the adjacent figure, calculate x so that the area of the colored surface would be equal to 16 cm^2 .



TEST

1 Complete. **(2 points)**

1º) 36 is the square of ... .

2º) 5 is the positive square root of ... .

3º) ... is the negative square root of 49 .

4º) 0.2 is a square root of ... .

2 Write without radicals.

$(\sqrt{51})^2$; $\sqrt{(-3)^2}$; $-\sqrt{\frac{1}{9}}$; $\sqrt{0.49}$; $-\sqrt{(17)^2}$;

$\sqrt{1-\frac{7}{16}}$; $\sqrt{(\pi-9)^2}$; $\sqrt{(\sqrt{3})^4}$; $\sqrt{1.21}$. **(3 points)**

3 Find a whenever possible ($a \geq 0$).

$\sqrt{a} = 9$; $\sqrt{a} = 1.3$; $\sqrt{a} = 1$; $\sqrt{a} = \frac{3}{5}$. **(2 points)**

4 Use your calculator to give the approximate value to the nearest 10^{-3} of the following :

$\sqrt{17}$; $+\sqrt{23}$; $\sqrt{\frac{4}{3}}$; $+\sqrt{315}$. **(2 points)**

5 Write in the form of $a\sqrt{b}$ ($b \geq 0$). **(2 points)**

$A = 0.7\sqrt{3} - 1.2\sqrt{3}$;

$B = +8\sqrt{19} + 2\sqrt{19} - 6\sqrt{19}$;

$C = 2\sqrt{x} + 4\sqrt{x} - 9\sqrt{x} + \sqrt{x}$ ($x \geq 0$) ;

$D = -5\sqrt{t} + 8\sqrt{t} - \sqrt{t}$ ($t \geq 0$).

6 Calculate and reduce.

$\sqrt{5}(\sqrt{5}+2)$; $(\sqrt{7}-6)(\sqrt{7}+6)$. **(2 points)**

7 Consider $x = \sqrt{21-y}$.

1º) Calculate x, if possible, for $y = -4$; $y = 20$; $y = 25$.

2º) Find the value of y knowing that $x = 4$.

8 Complete the following chart. **(3 points)**

x	5	8.2	– 1	+ 2.5	0
$\sqrt{(x-3)^2}$					
$\sqrt{(3-x)^2}$					

9 Using your calculator, find the measure of a square piece of land whose area is equal to 231.04 m^2 . **(1 point)**

5 PARALLELOGRAM

Objective

To know and to use the properties of a parallelogram.

CHAPTER PLAN

COURSE

1. Definition

2. Properties of a parallelogram

3. Conditions so that a quadrilateral would be a parallelogram

4. Exercise of construction

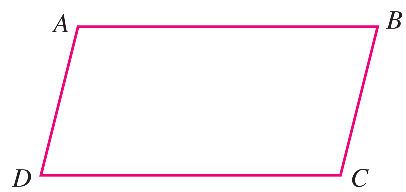
EXERCISES AND PROBLEMS

TEST

COURSE

1 DEFINITION

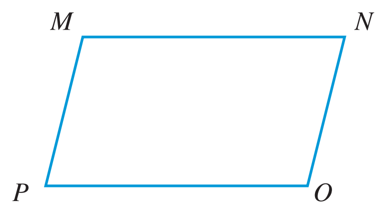
A **parallelogram** is a quadrilateral having its **opposite sides parallel**.



(AB) // (CD) and (AD) // (BC)

2 PROPERTIES OF A PARALLELOGRAM

Activity



MNOP is a quadrilateral having parallel opposite sides.

1º) **a)** Prove that the two triangles *MNO* and *MPO* are congruent.

b) Therefore, find in *MNOP*, the congruent sides and the equal angles.

2º) The diagonals $[MO]$ and $[NP]$ intersect at I .

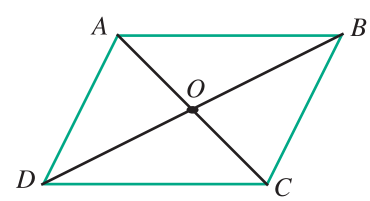
a) Prove that the two triangles *MIN* and *PIO* are congruent.

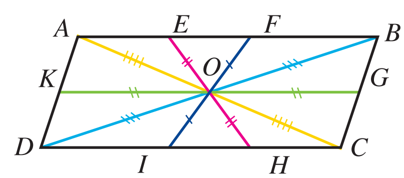
b) Deduce that I is the midpoint of $[MO]$ and $[PN]$.

Properties

In a parallelogram :

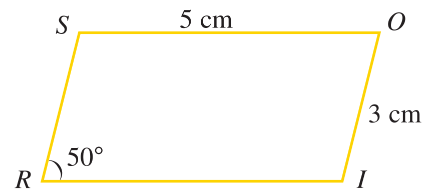
- The **opposite sides** are **congruent** ($AB = DC$ and $AD = BC$).
- The **opposite angles** are **equal** ($\widehat{DAB} = \widehat{DCB}$ and $\widehat{ADC} = \widehat{ABC}$).
- The **diagonals** intersect at **their midpoint** ($OA = OC$ and $OB = OD$).
- The parallelogram admits the intersecting point of its diagonals as a center of symmetry.





$OA = OC$; $OB = OD$; $OI = OF$;
$OH = OE$; $OK = OG$.

Application 1



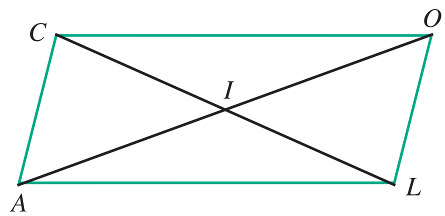
1º) **a)** Find the perimeter of the adjacent parallelogram *SOIR* .

b) Calculate its angles.

2º) **a)** Draw a segment [*MN*] and mark a point *I* not situated on (*MN*).

b) Construct the parallelogram *MINE* having [*MN*] as diagonal.

3º) Tell whether in each of the following cases the quadrilateral *COLA* is a parallelogram.



a) $CO = LA$ and $CA = LO$.

b) $\widehat{ACO} = \widehat{ALO}$ and $\widehat{CAL} = \widehat{COL}$.

c) $CA = LO$.

d) $IC = IO$.

3 CONDITIONS SO THAT A QUADRILATERAL WOULD BE A PARALLELOGRAM

In order for a quadrilateral to be a parallelogram, one of the following conditions should be verified :

- The **opposite sides** are **parallel**.
- The **opposite sides** are **congruent**..
- **Two opposite sides** are **parallel** and **congruent**.
- The **opposite angles** are **equal**.
- The **diagonals bisect** each **other**.

Application 2

ABC is any triangle. *E* and *F* are the midpoints of [*AB*] and [*AC*]. *H* is the symmetric of *E* with respect to *F*.

1º) Prove that the quadrilateral *AECH* is a parallelogram.

2º) Prove that *EHCB* is a parallelogram.

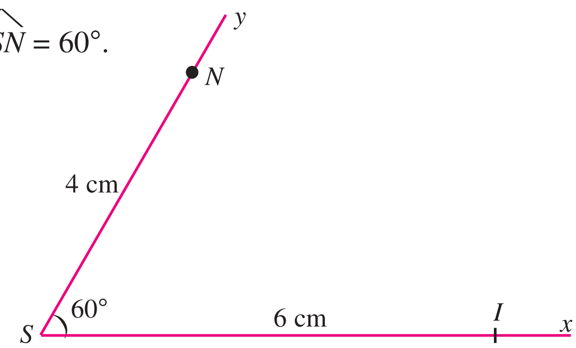
4 EXERCISE OF CONSTRUCTION

Consider the parallelogram *SIEN* to be constructed :

$SI = 6$ cm, $SN = 4$ cm and $\widehat{ISN} = 60°$.

Draw angle $\widehat{xSy} = 60°$.

Place the point I on $[Sx)$ such that $SI = 6$ cm and the point N on $[Sy)$ such that $SN = 4$ cm.



To determine the point E, one of the following methods may be used.

1st method

Draw from I the parallel to $[SN)$ and from N the parallel to $[SI)$; E is their intersecting point.

2nd method

Draw from I the parallel to $[SN)$. On this parallel place a length $IE = SN = 4$ cm.

3rd method

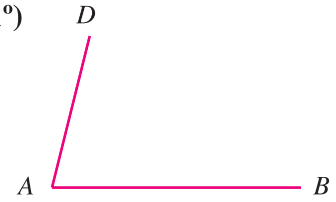
Determine the midpoint O of $[IN]$, then on $[SO]$ locate point E such that $OE = SO$.

EXERCISES AND PROBLEMS

Test your knowledge

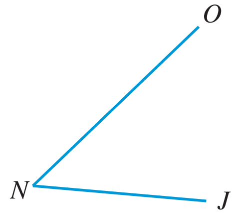
1 The construction of the parallelogram is incomplete. Complete the construction, justifying the method.

1º)



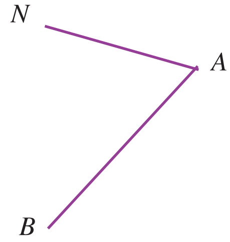
ABCD is a parallelogram.

2º)



NOIR is a parallelogram of center *J*.

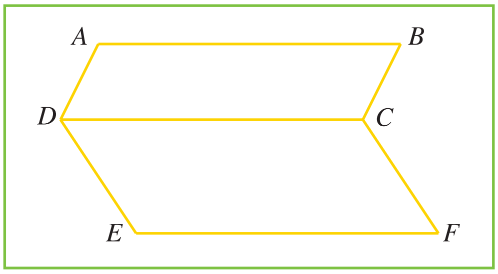
3º)



BIEN is a parallelogram of center *A*.

2 *ABCD* and *CDEF* are two parallelograms constructed on both sides of (*CD*).

Prove that *ABFE* is a parallelogram.



3 *MONT* is a parallelogram.

A is the midpoint of [*ON*] and *I* is the symmetric of *M* with respect to *A*.

1º) Prove that *MOIN* is a parallelogram.

2º) Prove that the points *T*, *N* and *I* are collinear.

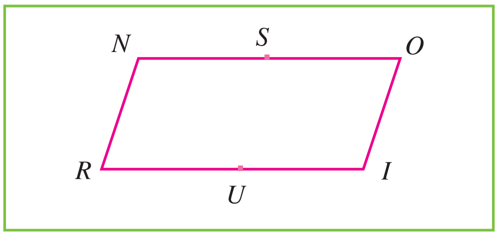
3º) Prove that *N* is the midpoint of [*TI*].

4 Quadrilateral *NOIR* is a parallelogram.
S is the midpoint of [*ON*] and *U* is the midpoint of [*RI*].

1º) Prove that (*SU*) // (*OI*).

2º) Prove that quadrilateral *SOUR* is a parallelogram.

3º) Name the common diagonal to *NOIR* and *SOUR*.
Then deduce that both parallelograms have the same center of symmetry.



5 Draw a segment $[BC]$ of length 6 cm.

1º) Construct a point A such that the distance from A to (BC) is equal to 5 cm.

2º) Construct the point D such that $ABDC$ is a parallelogram.

6 Construct a parallelogram $ABCD$ in each of the following cases.

1º) $AB = 6$ cm , $AD = 4$ cm and $\widehat{BAD} = 115°$.

2º) $AB = 5$ cm , $BC = 3$ cm and $AC = 6$ cm.

7 (d) and (d') are two lines intersecting at O. J is a point that belongs neither to (d), nor to (d').

Let K be the symmetric of O with respect to J.

The parallel to (d') passing through K cuts (d) at A, and the parallel to (d) passing through K cuts (d') at B.

Prove that J is the midpoint of $[AB]$.

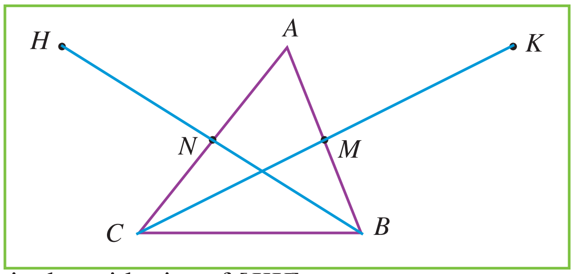
8 $ABCD$ is a parallelogram. E and F are respectively the orthogonal projections of A and C on line (BD).

1º) Prove that the triangles AED and BCF are congruent.

2º) Prove that $AECF$ is a parallelogram.

3º) Deduce that $[BD]$ and $[EF]$ have the same midpoint.

9 ABC is a triangle. M is the midpoint of $[AB]$ and N is the midpoint of $[AC]$. Consider K the symmetric of C with respect to M and H the symmetric of B with respect to N.



1º) Prove that $ACBK$ and $ABCH$ are two parallelograms.

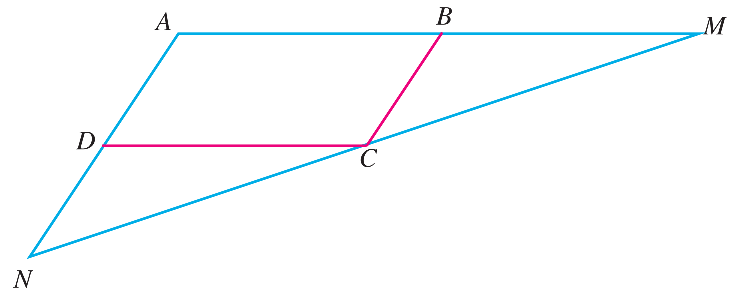
2º) Deduce that K, A and H are collinear and that A is the midpoint of $[KH]$.

10 M, O and N are three non-collinear points. I is the midpoint of $[OM]$ and J the midpoint of $[ON]$.

1º) Construct the parallelogram $JOIE$.

2º) Prove that the three points M, E and N are collinear and deduce that E is the midpoint of $[MN]$.

11 *ABCD* is a parallelogram.
Extend the side [*AB*] so that $BM = AB$.
Line (*MC*) cuts (*AD*) at *N*.

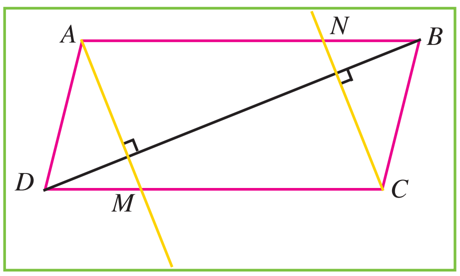


1º) Prove that the two triangles *BMC* and *DCN* are congruent.

2º) Prove that *BCND* is a parallelogram.

3º) Prove that $MN = 2BD$.

12 *ABCD* is a parallelogram. The perpendicular to (*BD*) passing through *A* cuts (*CD*) at *M*. The perpendicular to (*BD*) passing through *C* cuts (*AB*) at *N*.



1º) Prove that quadrilateral *AMCN* is a parallelogram.

2º) Deduce that the segments [*AC*], [*BD*] and [*MN*] have the same midpoint.

3º) Prove that *BMDN* is a parallelogram.

13 Answer by true or false.

1º) A parallelogram admits a center of symmetry which is the point of intersection of its diagonals.

2º) In the parallelogram *MNPQ*, [*MN*] and [*PQ*] are the diagonals.

3º) In the parallelogram *EFGH*, if $\widehat{HEF} = 70°$ then $\widehat{HGF} = 70°$.

4º) In a parallelogram, any two consecutive sides are congruent.

5º) The sum of the angles in a parallelogram is equal to 360°.

6º) In a parallelogram, any two opposite sides have the same perpendicular bisector.

For seeking

14 $ABCD$ is a parallelogram and M is the midpoint of $[AB]$. The lines (DM) and (BC) intersect at E.

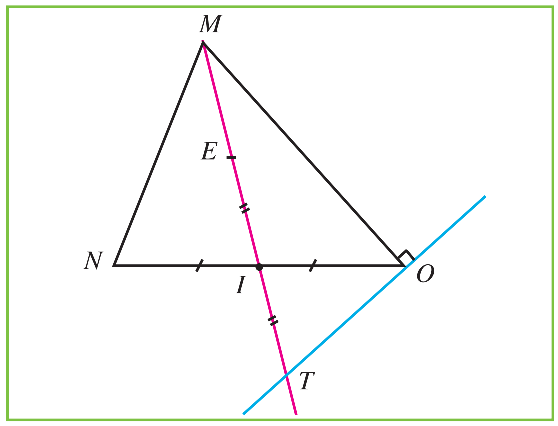
1º) Prove that the triangles AMD and BME are congruent.

2º) Prove that :

a) $BEAD$ is a parallelogram.

b) B is the midpoint of $[EC]$.

15 Consider a triangle MON . I is the midpoint of $[ON]$. The perpendicular from O to (MO) cuts (MI) at T. Let E be the symmetric of T with respect to I.



1º) What is the nature of quadrilateral $OTNE$? Justify.

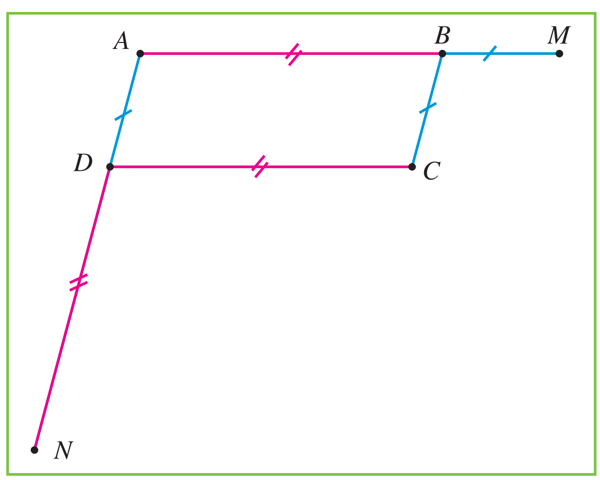
2º) Prove that (NE) and (MO) are perpendicular.

16 ABC is an isosceles triangle of vertex A. M is a point on $[BC]$. The parallel drawn from M to (AB) cuts $[AC]$ at E and the parallel drawn from M to (AC) cuts $[AB]$ at F.

1º) Prove that the triangle MEC is isosceles.

2º) Prove that the perimeter of $AEMF$ is equal to $2 \times AC$.

17 Extend respectively the sides $[AB]$ and $[AD]$ of a parallelogram $ABCD$ such that $BM = AD$ and $DN = AB$.

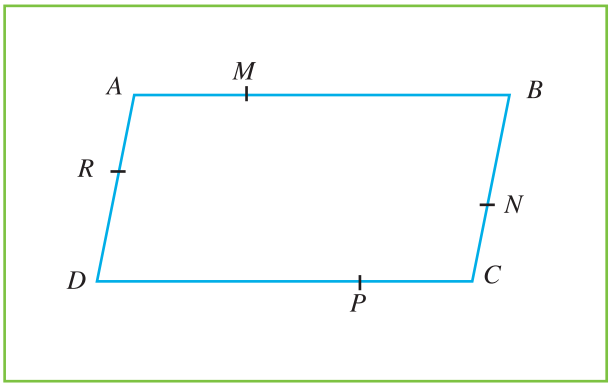


1º) Prove that the triangles BMC and DNC are isosceles.

2º) Prove that the points M, C and N are collinear.

18 On the sides AB, BC, CD and DA of a parallelogram $ABCD$ locate the points M, N, P and R such that :

$AM = BN = CP = DR$.

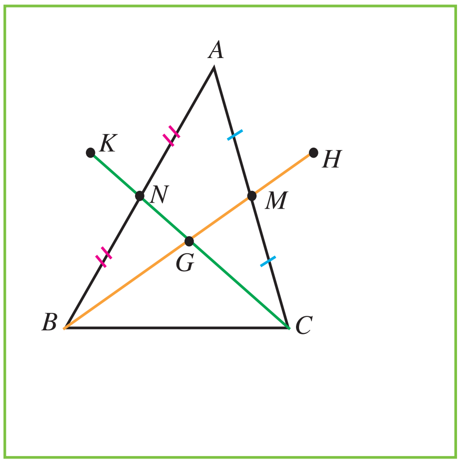


1º) Prove that $[RN]$ and $[BD]$ have the same midpoint.

2º) Prove that $[AC]$ and $[MP]$ have the same midpoint.

3º) Prove that $MNPR$ is a parallelogram.

19 G is the center of gravity of a triangle ABC, M is the midpoint of $[AC]$ and N the midpoint of $[AB]$. Consider H and K the symmetrics of G with respect to M and N respectively.



1º) Prove that $AGCH$ and $AGBK$ are two parallelograms.

2º) Prove that $BCHK$ is a parallelogram.

20 $ABCD$ is a parallelogram. The bisector $[Ax)$ of $\widehat{DAB}$ cuts (DC) at M. The bisector $[Cy)$ of $\widehat{DCB}$ cuts (AB) at N.

1º) Show that (Ax) is parallel to (Cy).

2º) Show that $AMCN$ is a parallelogram.

3º) Show that (DB) passes through the midpoint O of $[MN]$; deduce that $BMDN$ is a parallelogram.

TEST

1 **1º)** Construct a parallelogram *NOIR* knowing that :

$NO = 8$ cm , $NR = 6$ cm and $\widehat{ONR} = 140°$. **(2 points)**

2º) The bisectors of $\widehat{NRI}$ and $\widehat{RIO}$ intersect at H.

Prove that triangle *RHI* is right. **(2 points)**

2 *ABC* is a triangle. *I* is the midpoint of [*AB*] and *J* the midpoint of [*BC*].
Construct point *F* the symmetric of *A* with respect to *J*.

1º) Prove that *ACFB* is a parallelogram. **(2 points)**

2º) *D* is the midpoint of [*CF*]. **(2 points)**

Prove that *BICD* is a parallelogram.

3º) Prove that *ACFB* and *BICD* have the same center of symmetry. **(2 points)**

3 *MON* is a triangle. *A*, *B* and *C* are respectively the midpoints of [*MO*], [*ON*] and [*MN*].

Construct *I* the symmetric of *B* with respect to *A* and *J* the symmetric of *B* with respect to *C*.

1º) Prove that *BOIM* and *BNJM* are two parallelograms. **(3 points)**

2º) Prove that the points *I*, *M* and *J* are collinear and that *M* is the midpoint of [*IJ*].

(2 points)

4 *ABCD* is a parallelogram. Construct *M* the symmetric of *A* with respect to *B* and *N* the symmetric of *A* with respect to *D*.

1º) Prove that (*CM*) is parallel to (*DB*). **(3 points)**

2º) Prove that the points *M*, *C* and *N* are collinear and that *C* is the midpoint of [*MN*].

(2 points)

6

RATIONAL EXPRESSIONS (LITERAL FRACTIONS)

Objective

Perform calculations on literal fractions.

CHAPTER PLAN

COURSE

1. Rational expressions (Literal fractions)
2. Properties
3. Operations on literal fractions

EXERCISES AND PROBLEMS

TEST

1 RATIONAL EXPRESSIONS (LITERAL FRACTIONS)

Activity

Find the following, if possible :

1º) the numerical value of the expression $\frac{6}{x}$ for $x = 2$; $x = 5$; $x = 0$. What do you notice ?

2º) the numerical value of the expression $\frac{x+4}{x-2}$ for $x = 5$; $x = 4$; $x = 2$. What do you notice ?

Definition

A fraction containing variables, denoted by letters, is called a literal fraction. It is defined when its denominator is not zero.

Application 1

a, b and m are non-zero integers. Are $\frac{5a}{7b}$ and $\frac{4}{3m}$ non-zero integers ?

2 PROPERTIES

1º) Activity

Use your calculator to calculate $\frac{-2}{5}$; $\frac{2}{-5}$ and $-\frac{2}{5}$.

What do you notice ?

Rule

a and b are two non-zero numbers, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.

2º) Activity

Use your calculator to calculate $\frac{-3}{-8}$ and $\frac{3}{8}$. What do you notice ?

Rule

a and b are non-zero numbers, $\frac{-a}{-b} = \frac{a}{b}$.

3º) Activity

Use your calculator to calculate $\frac{-21}{15}$ and $\frac{-7}{5}$. What do you notice ?

Rule

Since a , b and k are three non-zero numbers, then $\frac{a \times k}{b \times k} = \frac{a}{b}$.

EXAMPLE

Since x and y are two non-zero numbers, then $\frac{x^2y}{4xy} = \frac{x}{4}$.

Application 2

1º) Compare the two literal fractions $\frac{-7x}{5y}$ and $\frac{7x}{-5y}$ then $\frac{-2m}{3m}$ and $\frac{2m}{3m}$ (x, y and m are three non-zero integers).

2º) Simplify (a, b, x, y and z are non-zero integers). $\frac{3a}{9b}$; $\frac{-25xy}{35xy}$; $\frac{-7x^2y}{-21xyz}$.

3 OPERATIONS ON LITERAL FRACTIONS

If a, b, c and d are integers, then :

⊙ $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ $(b \neq 0)$.

⊙ $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$ $(b \neq 0)$.

⊙ $\frac{a}{b} + \frac{c}{d} = \frac{a\,.\,d}{b\,.\,d} + \frac{b\,.\,c}{b\,.\,d} = \frac{a\,.\,d + b\,.\,c}{b\,.\,d}$ $(b \neq 0$ and $d \neq 0)$.

⊙ $\frac{a}{b} - \frac{c}{d} = \frac{a\,.\,d}{b\,.\,d} - \frac{b\,.\,c}{b\,.\,d} = \frac{a\,.\,d - b\,.\,c}{b\,.\,d}$ $(b \neq 0$ and $d \neq 0)$.

⊙ $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ $(b \neq 0$ and $d \neq 0)$.

⊙ $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a\,.\,d}{b\,.\,c}$ $(b \neq 0$, $c \neq 0$ and $d \neq 0)$.

⊙ $a \times \frac{b}{c} = \frac{a}{1} \times \frac{b}{c} = \frac{a\,.\,b}{c}$ $(c \neq 0)$.

⊙ $-a \times \frac{b}{c} = \frac{-a}{1} \times \frac{b}{c} = -\frac{a\,.\,b}{c}$ $(c \neq 0)$.

EXAMPLES

(All the variables are non-zero integers).

1º) $\frac{2}{m} + \frac{t}{n} = \frac{2n + tm}{mn}$ **2º)** $\frac{-2x}{5y} \times \frac{-10y}{6x} = \frac{20xy}{30xy} = \frac{2}{3}$ **3º)** $\frac{m}{n} \div \frac{n}{m} = \frac{m}{n} \times \frac{m}{n} = \frac{m^2}{n^2}$.

Application 3

Calculate (all the variables are non-zero integers).

1º) $\frac{2a}{x} - \frac{3y}{5x}$ **2º)** $\frac{5}{m} - \frac{3}{2n}$ **3º)** $\frac{2a}{3} + \frac{3a}{2}$ **4º)** $\frac{x}{-y} \times \frac{-zy}{xt}$.

EXERCISES AND PROBLEMS

Test your knowledge

(In all of the exercises, the variables in the denominators are non-zero integers, and the variables in the numerators are integers).

1 Find the numerical value of each of the following literal fractions.

1º) $\frac{x}{3y}$ where $x = 2$ and $y = 6$. **2º)** $\frac{2a}{5b}$ where $a = -10$ and $b = -4$.

3º) $\frac{x+1}{x-2}$ where $x = 5$.

2 Answer by true or false.

1º) $-\frac{2}{t}$ is a literal fraction. **2º)** $\frac{a \, . \, b}{a \, . \, c}$ is not a literal fraction. **3º)** $\frac{-3a}{5b} = -\frac{3a}{5b}$.

4º) $\frac{5}{-a} = \frac{-5}{a} = +\frac{5}{a}$. **5º)** $\frac{3}{m} + \frac{10}{n} = \frac{3 + 10}{m \, . \, n}$. **6º)** $\frac{a}{5} + \frac{b}{5} = \frac{a + b}{5}$.

7º) $\frac{x}{y} = -0{,}6$ where $x = 3$ and $y = -5$. **8º)** $\frac{a^2 b}{b^2 a} = \frac{a}{b}$. **9º)** $\frac{-m}{-n} = -\frac{m}{n}$.

3 Simplify the following literal fractions.

$\frac{5a}{10b}$; $\frac{-21xyz}{35xyt}$; $\frac{m \, . \, n^2}{m \, . \, n \, . \, f}$; $\frac{3mn}{2em}$; $\frac{6c^2}{24bc}$; $\frac{12amx^2}{4a^2mx}$.

4 Calculate the following.

$\frac{a}{4} + \frac{b}{4}$; $\frac{6x}{y} + \frac{-5x}{-3y}$; $\frac{x}{5} + \frac{-x}{10}$; $\frac{2x}{5} - \frac{7x}{15}$.

5 Calculate and simplify, if possible.

1º) $\frac{2a}{3} \times \frac{3b}{-4}$. **2º)** $\frac{-9x}{2y} \times \frac{y}{-3x}$. **3º)** $-4xy \times \frac{y}{x}$.

4º) $\frac{x}{y} \div \frac{x}{y}$. **5º)** $\frac{x}{y} \div \frac{y}{x}$. **6º)** $\frac{3a}{b} \div \frac{a}{2}$.

7º) $\frac{3a}{5b} \div \frac{6a}{-10}$. **8º)** $\frac{8m^2}{12} \div 4m^2$. **9º)** $-6ay^2 \div \frac{2ay}{5}$.

For seeking

6 Calculate the following and simplify, if possible.

1º) $\frac{2a}{3} + \frac{3a}{2} - \frac{5a}{6}$. **2º)** $\frac{b}{a} + \frac{a}{b}$. **3º)** $5a + \frac{b}{c}$.

4º) $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$. **5º)** $\frac{a-3}{3} + \frac{a+5}{6} + \frac{a-4}{9}$. **6º)** $\frac{4x}{x-2} - \frac{8}{-2+x}$ $(x \neq 2)$.

7 Simplify then calculate, if possible.

$\frac{-10x^2}{9y^2} \times \frac{27y^3}{25x^3}$; $\frac{8m^2}{12m} - 4m$; $\frac{3a^2b^2}{-9a^3b}$; $\frac{c^2 - cd}{bc}$; $\frac{-15k\ell}{20mk\ell}$.

TEST

All the variables in the denominators are non-zero integers.

1 Answer by true or false. **(9 points)**

1º) $\dfrac{-5b}{7c} = \dfrac{5b}{-7c}$.

2º) $\dfrac{2}{a} + \dfrac{-b}{-a} = \dfrac{2+b}{a}$.

3º) $\dfrac{e}{f} + \dfrac{f}{e} = 1$.

4º) $\dfrac{x}{y} = -0.14$ for $x = -7$ and $y = 5$.

5º) $\dfrac{mn^2}{nm^2} = \dfrac{m}{n}$.

6º) $\dfrac{3}{m} + \dfrac{7}{n} = \dfrac{3n+7m}{mn}$.

2 Simplify the following literal fractions. **(3 points)**

1º) $\dfrac{12xyz}{-30x^2y}$.

2º) $\dfrac{-3ab^2c^3}{9\,a^2bc^2}$.

3º) $\dfrac{-8\,x^2y}{-24\,xy^2}$.

3 Perform the following operations and simplify, if possible. **(8 points)**

1º) $\dfrac{x}{15} + \dfrac{-y}{5}$.

2º) $\dfrac{-8a}{3b} \times \dfrac{6b}{16a}$

3º) $\dfrac{16m^2}{9} \div \dfrac{8m^3}{-3}$.

4º) $\dfrac{t-2}{5} + \dfrac{t+3}{4} - \dfrac{t-8}{10}$.

7 COMPOUND FRACTION

Objectives

1. To know the reciprocal of a non-zero number.

2. To know how to divide two fractions.

CHAPTER PLAN

COURSE

1. Reciprocal of a non-zero number

2. Quotient of $\frac{a}{b}$ by $\frac{c}{d}$

EXERCISES AND PROBLEMS

TEST

COURSE

1 RECIPROCAL OF A NON-ZERO NUMBER

Activity

1º) Calculate the following products :

$2 \times \frac{1}{2}$; $\frac{-3}{5} \times \frac{5}{-3}$; $\frac{x}{y} \times \frac{y}{x}$ ($x \neq 0$ and $y \neq 0$) ; $a \times \frac{1}{a}$ ($a \neq 0$).

2º) Complete : $4 \times ... = 1$; $\frac{9}{-7} \times ... = 1$; $\frac{-3}{4} \times ... = 1$.

3º) Is there a number whose product by 0 gives 1 ?

Definition

When the **product** of two numbers is equal to **1**, one of these numbers is **the reciprocal** of the other.

⊙ a is a non-zero number.

$a \times \frac{1}{a} = 1$; therefore the reciprocal of a is $\frac{1}{a}$ and the reciprocal of $\frac{1}{a}$ is a.

We write : $\boxed{\dfrac{1}{\frac{1}{a}} = a}$

⊙ a and b are two non-zero numbers. Since $\frac{a}{b} \times \frac{b}{a} = 1$, therefore both fractions $\frac{a}{b}$ and $\frac{b}{a}$ are the reciprocals of each other.

$$\boxed{\dfrac{1}{\frac{a}{b}} = \frac{b}{a}}$$

⊙ No number multiplied by 0 gives 1; therefore, 0 does not have a reciprocal.

EXAMPLES

⊙ The reciprocal of 3 is $\frac{1}{3}$.

⊙ The reciprocal of $\frac{-1}{5}$ is $\dfrac{1}{\frac{-1}{5}} = \frac{5}{-1} = -5$.

⊙ The reciprocal of $\frac{5}{7}$ is $\dfrac{1}{\frac{5}{7}} = \frac{7}{5}$.

2 QUOTIENT OF $\frac{a}{b}$ BY $\frac{c}{d}$

Activity

Use the calculator to calculate : $\frac{2}{5} \div \frac{3}{4}$ and $\frac{2}{5} \times \frac{4}{3}$. What do you notice ?

Rule

a, b, c and d are numbers such that $b \neq 0$, $c \neq 0$ and $d \neq 0$.

To divide two fractions is to multiply the first by the reciprocal of the second.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}$$

The form $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ is called **compound fraction.**

Remarks

⊙ $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \dfrac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}$ ($b \neq 0$ and $c \neq 0$).

⊙ $a \div \frac{b}{c} = \frac{a}{1} \div \frac{b}{c} = \dfrac{\frac{a}{1}}{\frac{b}{c}} = \frac{a}{1} \times \frac{c}{b} = \frac{ac}{b}$ ($b \neq 0$ and $c \neq 0$).

EXAMPLES

⊙ $2 \div 3 = 2 \times \frac{1}{3} = \frac{2}{3}$.

⊙ $\frac{3}{5} \div 2 = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$.

⊙ $\frac{9}{5} \div \frac{-4}{7} = \frac{9}{5} \times \frac{7}{-4} = -\frac{63}{20}$.

⊙ $0 \div \frac{5}{7} = 0 \times \frac{7}{5} = 0$.

⊙ $\dfrac{\frac{3}{7}}{\frac{2}{5}} = \frac{3}{7} \div \frac{2}{5} = \frac{3}{7} \times \frac{5}{2} = \frac{15}{14}$.

⊙ $-\frac{14}{25} \div \frac{7}{6} = \frac{-14}{25} \times \frac{6}{7} = \frac{-2 \times 7 \times 6}{25 \times 7} = \frac{-12}{25}$.

⊙ $\frac{8}{-15} \div \frac{4}{-3} = \frac{8}{-15} \times \frac{-3}{4} = \frac{2 \times 4 \times (-3)}{(-3) \times 5 \times 4} = \frac{2}{5}$.

Application

Calculate and simplify the following compound fractions.

$\dfrac{\frac{3}{4}}{\frac{9}{5}}$; $\frac{32}{21} \div \frac{24}{15}$; $\dfrac{\frac{x}{3}}{\frac{5x}{9}}$ $(x \neq 0)$.

EXERCISES AND PROBLEMS

Test your knowledge

1 Complete the following chart, if possible.

a	$\frac{3}{2}$		0		
Opposite of a		-4			
Reciprocal of a				$\frac{6}{11}$	-0.4

2 Complete.

1°) $\frac{13}{17} \times \ldots = 1$.

2°) $\ldots \times \frac{-7}{18} = 1$.

3°) $\frac{x}{2y} \times \ldots = 1 \quad (x \neq 0 \quad \text{and} \quad y \neq 0)$.

4°) $-a \times \ldots = 1 \quad (a \neq 0)$.

3 Write in the form of $\frac{a}{b}$ where a and b are two integers.

1°) $\dfrac{\frac{1}{8}}{7}$

2°) $\dfrac{1}{\frac{-3}{4}}$

3°) $\dfrac{-\frac{1}{3}}{3}$

4°) $\dfrac{\frac{5}{9}}{\frac{8}{3}}$

5°) $\dfrac{9}{\frac{1}{3}}$

6°) $\dfrac{32}{\frac{-1}{8}}$

7°) $\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}$

8°) $\left(\frac{-4}{15}\right) \div \left(\frac{-7}{-3}\right)$

9°) $\dfrac{2-\frac{3}{4}}{1+\frac{9}{2}}$.

4 **1°)** Calculate, knowing that $a = \frac{5}{4}$ and $b = -\frac{2}{3}$.

$\dfrac{\frac{1}{a}}{\frac{1}{b}}$; $\dfrac{1}{\frac{a}{b}}$; $\dfrac{b}{a}$.

2°) Then compare : $\dfrac{b}{a}$ and $\dfrac{\frac{1}{a}}{\frac{1}{b}}$.

5 Complete the following chart.

Non-calculated quotient	$\frac{-3}{28} \div \frac{9}{7}$	$-9 \div \frac{-21}{9}$	$\frac{72}{7} \div (-18)$	$\frac{0.6}{0.8} \div \frac{5}{4}$	$\frac{-98}{3} \div 1.2$
Sign of the quotient					
Calculated quotient in the form of an irreducible fraction					

6 Write in the form of an irreducible fraction.

$$X = \frac{2 + \frac{2}{3}}{2 - \frac{2}{3}} \quad ; \quad Y = \frac{7 - \frac{1}{2}}{7 + \frac{3}{4}} \quad ; \quad Z = \frac{\frac{1}{5} - \frac{1}{4}}{\frac{1}{4} - \frac{1}{5}} .$$

7 Answer by true or false.

1°) The reciprocal of 4 is 0.25 .

2°) The reciprocal of $\frac{1}{8}$ is $-\frac{1}{8}$.

3°) The reciprocal of $-\frac{3}{7}$ is $\frac{7}{3}$.

4°) To divide by $\frac{1}{5}$ is to multiply by 5.

5°) To divide by 0.5 is to multiply by 2.

6°) $\frac{a}{b} \div \frac{b}{a} = 1$ $(a \neq 0$ and $b \neq 0)$.

7°) If a number is positive, then its reciprocal is negative.

8°) The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ $(a \neq 0$ and $b \neq 0)$.

For seeking

8 Solve the following equations.

1°) $6a = -3$.

2°) $-\frac{1}{3}x = 2$.

3°) $b \div 3 = \frac{6}{7}$.

4°) $-\frac{4}{5} \cdot y = \frac{8}{15}$.

5°) $z \div \left(\frac{-35}{3}\right) = \frac{18}{7}$.

6°) $\frac{2}{7} \times x = \frac{3}{14}$.

7°) $\frac{3}{4}x = \frac{5}{8}$.

8°) $\frac{x}{2} = \frac{45}{10}$.

9 Calculate.

$$A = \frac{3+\frac{2}{5}+\frac{3}{4}}{3+\frac{1}{5}-\frac{3}{4}} \; ; \; B = \frac{4+0.5+7.5}{\frac{9}{4}-\frac{4}{3}} \; ;$$

$$C = \frac{0.2-\frac{1}{10}+0.02}{\frac{1}{4}+\frac{1}{100}-0.04} \, .$$

10 Calculate the numerical value of $A = \frac{a \, . \, b}{c}$ knowing that :

$a = \frac{2}{3}$, $b = \frac{7}{4}$ and $c = \frac{5}{6}$.

11 x is a non-zero natural number.

Simplify : $A = \dfrac{2+\frac{1}{x}+\frac{1}{2x}}{\frac{1}{2}+\frac{1}{x}}$.

12 Consider $A = a^2 - b^2$, $B = a \, . \, b$ and $C = \frac{a}{b} - \frac{b}{a}$, where a and b are two non-zero integers.

1°) Compare C and $\frac{A}{B}$.

2°) Find the numerical value of $C + \frac{A}{B}$, when $a = 5$ and $b = -4$.

13 Calculate and reduce.

$$A = \frac{1+\frac{1}{1+\frac{1}{3}}}{\frac{1}{1+\frac{1}{3}}} \; ; \; B = \frac{2+\frac{1}{2}}{2+\frac{2+\frac{1}{2}}{1+\frac{1}{2}}} .$$

14 Solve the following equation.

$$\frac{x}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}} = 1 \, .$$

15 The area of a triangle ABC is 100 cm^2 . The altitude relative to $[BC]$ measures $\frac{5}{7}$ dm. Calculate the length of $[BC]$.

16 A meadow of $\frac{4}{5}$ hm^2 is divided into 12 equal pieces.

What is the area of each piece ?

17 A laborer needs 14 days to do $\frac{7}{15}$ of a labor.

What part of the labor does he do per day ?

18 To cross a street 16 m(meters) wide, how many steps of $\frac{4}{7}$ m(meter) should be made?

19 The $\frac{3}{4}$ of my savings are equal to 150 000 L.L. How much money did I save ?

20 A merchant has divided a piece of linen cloth of 57 meters into pieces of $\frac{19}{4}$ meters and has sold them for \$ 18 the meter.
Find the number of pieces and the selling price of each piece.

TEST

1 Answer by true or false. **(6 points)**

1º) The reciprocal of 5 is 0.2.

2º) The reciprocal of the opposite of -4 is $-\dfrac{1}{4}$.

3º) To divide by $\dfrac{1}{4}$ is to multiply by 4.

4º) $\dfrac{\frac{3}{2}}{\frac{-6}{8}} = -2$.

5º) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \times \dfrac{d}{c}$ (with b, c and d non-zero).

6º) The reciprocal of $\dfrac{x}{y}$ is $-\dfrac{x}{y}$ ($x \neq 0$ and $y \neq 0$).

2 Write in the form of an irreducible fraction. **(4 points)**

1º) $A = \dfrac{3 + \frac{3}{5}}{\frac{3}{5} - 3}$ **2º)** $B = \dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$.

3 Calculate the numerical value of $X = \dfrac{x \,.\, y}{z \,.\, t}$ knowing that : **(2 points)**

$x = \dfrac{12}{5}$, $y = \dfrac{15}{8}$, $z = \dfrac{3}{2}$ and $t = -3$.

4 Consider $M = x^2 + y^2$, $P = x \,.\, y$ and $Q = \dfrac{x}{y} + \dfrac{y}{x}$, where x and y are two non-zero integers. **(4 points)**

1º) Compare Q and $\dfrac{M}{P}$.

2º) Find the numerical value of $\dfrac{M}{P} + 5Q$, where $x = 3$ and $y = -2$.

5 The distance between cities A and B is 105 km. A cyclist needs three hours to cover the $\dfrac{4}{5}$ of this distance.

1º) Calculate, in km, the distance covered by the cyclist.

2º) Find the average speed of the cyclist. **(4 points)**

8 SPECIAL PARALLELOGRAMS

Objective

To know the properties of a rectangle, a rhombus, a square.

CHAPTER PLAN

COURSE

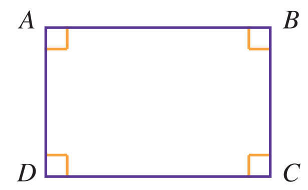
1. Rectangle
2. Rhombus
3. Square
4. Summary
5. Constructions

EXERCISES AND PROBLEMS

TEST

1 RECTANGLE

1. Definition



A **rectangle** is a quadrilateral having **four right angles.**

Therefore, a rectangle is a special parallelogram.

2. Activity

ABCD is a rectangle of center O .

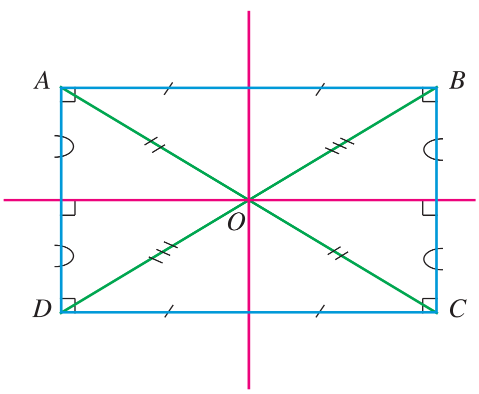
1º) Prove that the two triangles *ABC* and *ABD* are congruent. Deduce that $AC = BD$.

2º) Therefore, prove that $OA = OB = OC = OD$.

3º) Draw the perpendicular bisector (xy) of $[AB]$.
a) (xy) passes through O . Justify your answer.
b) (xy) is also the perpendicular bisector of $[CD]$. Justify your answer.

4º) The line (xy) is an axis of symmetry in the figure. Can you draw another one ? Which one ?

Properties



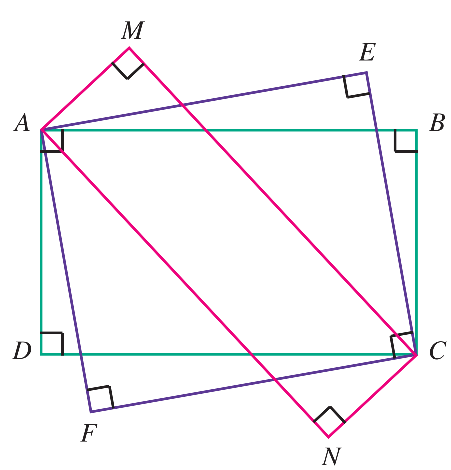
In a rectangle :

1º) The **opposite sides** are **parallel** and **congruent** (Since the rectangle is a parallelogram).

2º) The **diagonals** are **congruent** and **they intersect in their midpoint,** which is the center of symmetry of the rectangle.

3º) **Two opposite sides** have the **same perpendicular bisector,** which is an **axis of symmetry** of the figure. Therefore, a rectangle admits two axes of symmetry.

Application 1



ABCD, *AECF* and *AMCN* are three rectangles.

1º) Prove that $BD = MN = EF$.

2º) Prove that the midpoint O of $[AC]$ is the center of symmetry of the three rectangles.

3. Conditions so that a quadrilateral would be a rectangle.

A quadrilateral is a **rectangle if one of the following conditions** is **verified** :

- It has **four right angles** (by definition) (It is enough to have three).
- It is a **parallelogram with one right angle**.
- It is a **parallelogram having congruent diagonals**.

Application 2

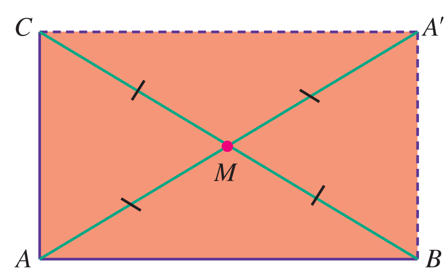
OAB is an isosceles triangle of vertex *O*. *C* and *D* are, respectively, the symmetrics of *A* and *B* with respect to *O*.

Prove that quadrilateral *ABCD* is a rectangle.

4. Properties of the median relative to the hypotenuse in a right triangle.

Statement 1

In any right triangle, the measure of the median relative to the hypotenuse is equal to half of the measure of the hypotenuse.



Given	***Required to prove***
$\widehat{BAC} = 90°$	$AM = \dfrac{BC}{2}$
$MB = MC$	

Proof

Consider A' the symmetric of A with respect to M.

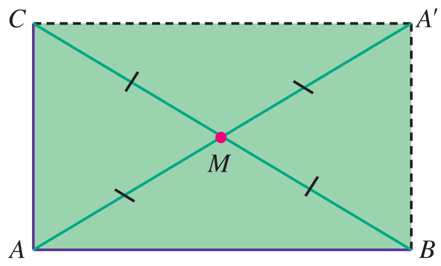
The quadrilateral $ABA'C$, having its diagonals intersecting at their midpoint, is a parallelogram.

Since angle A is right, then this parallelogram is a rectangle. Therefore its diagonals are congruent : $AA' = BC$.

Therefore : $AM = \dfrac{AA'}{2} = \dfrac{BC}{2}$.

Statement 2

In a triangle ABC, if the median from A is equal to half of $[BC]$, then this triangle is right at A .



Given	***R.T.P.***
$MB = MC$	$\widehat{BAC} = 90°$
$AM = \frac{BC}{2}$	(ABC is right at A).

Proof

Consider A' the symmetric of A with respect to M.

$MA = MA'$ and $MB = MC$, therefore, the quadrilateral $ABA'C$ is a parallelogram.

But $AM = \frac{BC}{2}$ hence $2\ AM = BC$; therefore $AA' = BC$.

The parallelogram $ABA'C$ is a rectangle since it has congruent diagonals.

Consequently $\widehat{BAC} = 90°$ and therefore triangle ABC is right at A.

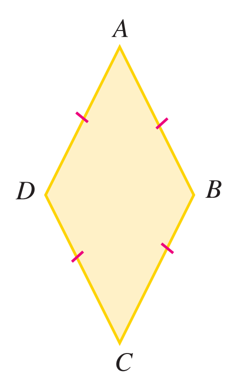
Application 3

1°) $[EF]$ is the diameter of a circle of center O and of radius R. G is a point of this circle, distinct from E and F.
Prove that GEF is a right triangle.

2°) TEN is a right triangle at T such that $\widehat{TEN} = 30°$. $[TO]$ is the median relative to $[NE]$.
Calculate the angles of the obtained figure.

2 RHOMBUS

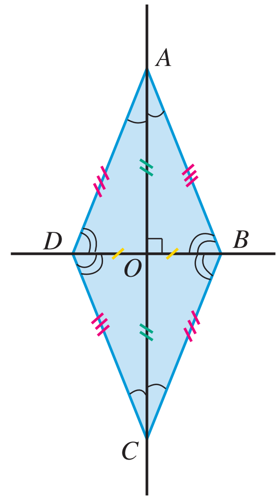
1. Definition



A **rhombus** is a quadrilateral having **congruent sides.**

Therefore, a rhombus is a special parallelogram.

2. Properties



In a rhombus :

1º) The **opposite sides are parallel** (it is a parallelogram).

2º) The **diagonals intersect at their midpoint, and they are perpendicular.** The support of one is the perpendicular bisector of the other.

3º) The **supports of the diagonals are two axes of symmetry** of the figure, and their common point is the center of symmetry.

4º) The **diagonals are the bisectors of the angles.**

3. Conditions so that a quadrilateral would be a rhombus.

A quadrilateral is a **rhombus if one of the following conditions is verified** :

- It has **four congruent sides** (definition).
- It is a **parallelogram having two consecutive congruent sides.**
- It is a **parallelogram having perpendicular diagonals.**.
- It is a **parallelogram having the diagonals the bisectors of the angles**.

Application 4

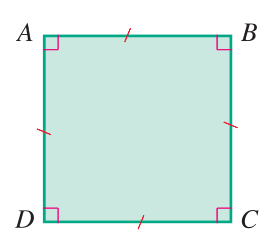
ABC is an isosceles triangle of vertex *A*. *M* is the midpoint of [*BC*]. *D* is the symmetric of *A* with respect to *M*.

Prove that quadrilateral *ABCD* is a rhombus.

3 SQUARE

1. Definition

A **square** is a quadrilateral having **four right angles** and **four congruent sides.**



2. Activity

ABCD is a square.

1º) Is *ABCD* a rectangle ? Justify.

2º) Is *ABCD* a rhombus ? Justify.

Properties

A square is at the same time a rectangle and a rhombus. Therefore, it has the properties of a rectangle and of a rhombus.

3. Conditions so that a quadrilateral would be a square

1º) A quadrilateral is a **square** if it has **four right angles and four congruent sides.**

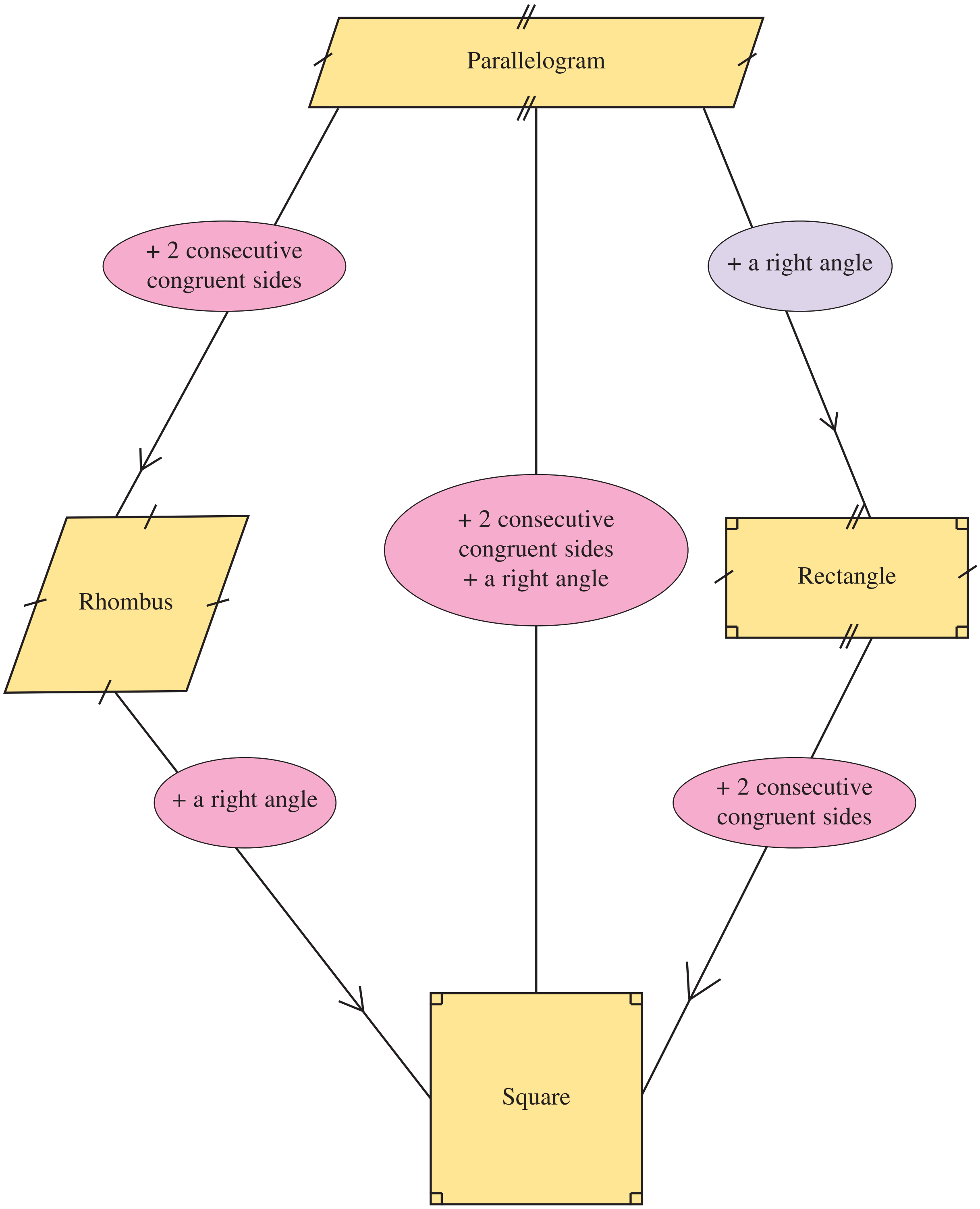
2º) A **quadrilateral** is a **square** if it is, **at the same time, a rectangle and a rhombus.**

Application 5

AOB is a right isosceles triangle of vertex *O*. [*OH*] is the altitude relative to [*AB*]. *P* is the symmetric of *H* with respect to (*OA*).

Prove that *OHAP* is a square.

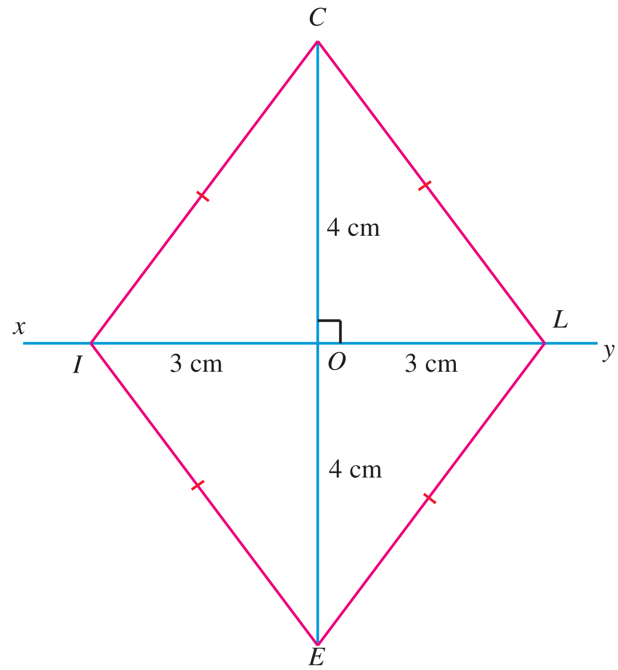
4 SUMMARY



Parallelogram
+ 2 consecutive congruent sides
+ a right angle
+ 2 consecutive congruent sides + a right angle
Rhombus
Rectangle
+ a right angle
+ 2 consecutive congruent sides
Square

5 CONSTRUCTIONS

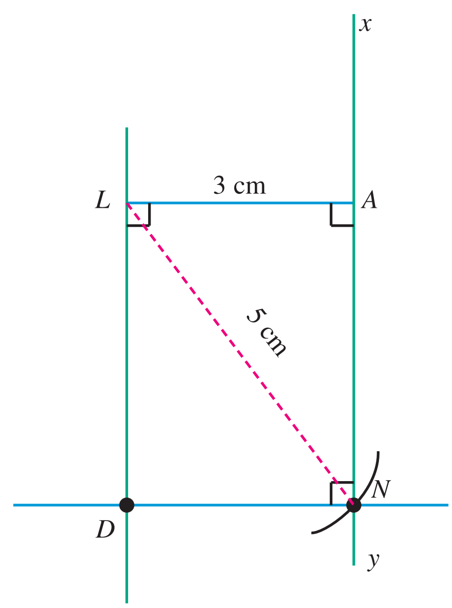
1º) Construct a rhombus *CIEL* such that $CE = 8$ cm and $LI = 6$ cm.



- In a rhombus, the diagonals are perpendicular and they intersect at their midpoint.
  Draw $[CE] = 8$ cm.
  From O, the midpoint of $[CE]$, draw (xy) perpendicular to (CE).
  On (xy), place the points I and L such that $OI = OL = 3$ cm.
  CIEL is the required rhombus.

2º) Construct a rectangle *LAND*, such that $LA = 3$ cm and $LN = 5$ cm.

- Draw $[LA] = 3$ cm.
  LAND, being a rectangle, hence $\widehat{LAN} = 90°$.
  Therefore, N belongs to the line (xy) perpendicular at A to(AL).
  Since $LN = 5$ cm, then N belongs also to the circle (C) of center L and of radius = 5 cm.
  N is the intersection of (xy) and of the circle (C).
  The perpendiculars at N to (xy) and at L to (LA) intersect at point D.
  LAND is the required rectangle.



Remark

The circle (C) cuts (xy) at another point N' ; therefore, there is a second rectangle $LAN'D'$.

EXERCISES AND PROBLEMS

Test your knowledge

1 [NI] and [OR] are two diameters of a circle (C). What is the nature of quadrilateral $NOIR$?

2 ABC is a triangle. [AM] is the altitude segment relative to [BC]. O is the midpoint of [AC] and N is the symmetric of M with respect to O.

Prove that $AMCN$ is a rectangle.

3 $ABCD$ is a rectangle of center O. P and Q are the feet of the perpendiculars drawn from O to [AB] and [BC] respectively.

1º) Prove that P is the midpoint of [AB] and Q is the midpoint of [BC].

2º) Prove that quadrilateral $OPBQ$ is a rectangle. Deduce that :

$OP = \frac{1}{2} BC$ and $OQ = \frac{1}{2} AB$.

3º) Prove that the perimeter of $ABCD$ is the double of the perimeter of $OPBQ$.

4 ABC is a triangle. I is the midpoint of [BC]. [CH] is the altitude relative to [AB] and [BP] is the altitude relative to [AC].

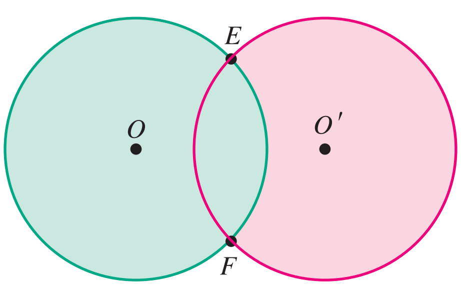
Prove that triangle HIP is isosceles.

5 OIE is an isosceles triangle of vertex O. Consider F the symmetric of I with respect to O.

Prove that (FE) and (IE) are perpendicular.

6 Two circles of centers O and O' and of equal radii intersect at E and F.

What is the nature of quadrilateral $OEO'F$?

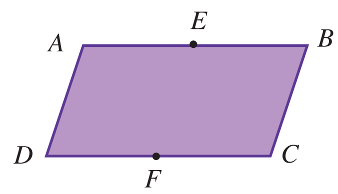


7 *ABCD* is a parallelogram such that $AB = 2\ AD$.

E is the midpoint of $[AB]$ and *F* is the midpoint of $[CD]$.

1º) Prove that *AEFD* and *BCFE* are rhombuses.

2º) Prove that triangle *AFB* is a right triangle.



8 *ABC* is an isosceles triangle of vertex *A*. *D* is the symmetric of *A* with respect to (BC) and *E* the symmetric of *A* with respect to *B*.

1º) Prove that *ACDB* is a rhombus.

2º) Prove that *BCDE* is a parallelogram.

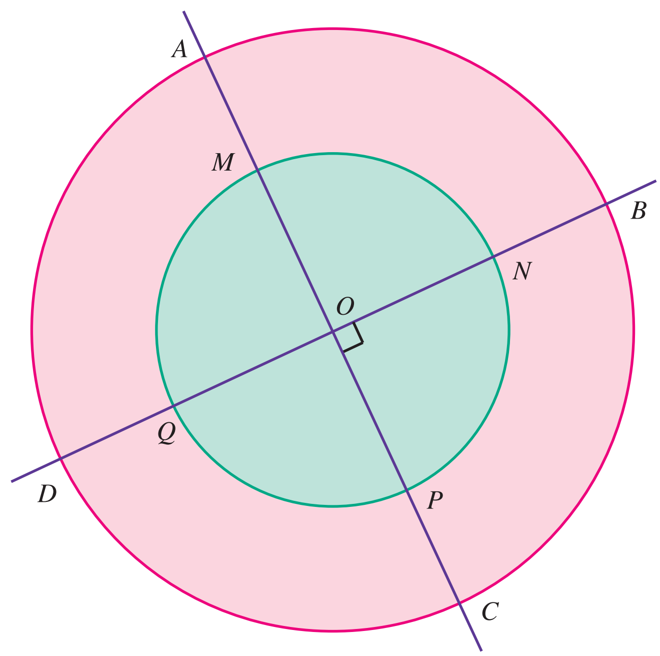
9 The adjacent figure shows two circles of the same center *O* with two perpendicular diameters.

1º) Name and draw two rhombuses.

Justify.

2º) Name and draw two squares.

Justify.



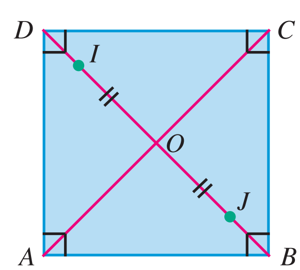
10 *ROLA* is a rhombus. The circle of diameter $[RL]$ cuts line (OA) at *I* and *E*.

What is the nature of quadrilateral *LIRE* ?

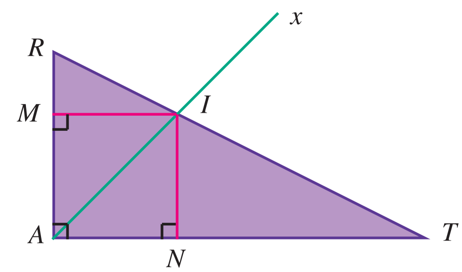
11 *ABCD* is a square of center *O*.

I and *J* are two points on (BD) symmetric to each other with respect to *O*.

What is the nature of quadrilateral *JAIC* ?



12 RAT is a right triangle at A.



The bisector $[Ax)$ of angle $\widehat{RAT}$ cuts $[RT]$ at I. Consider M and N the orthogonal projections of I on $[AR]$ and $[AT]$ respectively.

1º) Prove that $IM = IN$.

2º) Prove that $AMIN$ is a square.

13 Construct a rectangle $HADI$ such that $HA = 2$ cm and $HI = 3$ cm.

14 Construct a rectangle $MATH$ such that $AM = 4$ cm and $MT = 5$ cm.

15 Construct a rhombus $MONA$ of a side = 5 cm and such that $\widehat{OMN} = 60°$.

16 Construct a rhombus $SOIR$ such that $SI = 6$ cm and $OR = 4$ cm.

17 Construct a square $MONI$ whose diagonal $[OI]$ measures 6 cm.

18 Answer by true or false.

1º) A rectangle is a parallelogram having a right angle.

2º) Two consecutive sides of a rectangle are perpendicular.

3º) The diagonals of a rectangle are perpendicular.

4º) The diagonals of a rectangle are congruent.

5º) In a rhombus, the support of a diagonal is the perpendicular bisector of the other diagonal.

6º) The diagonals of a rhombus are congruent.

7º) The sides of a rhombus are congruent.

8º) A square is a rhombus having a right angle.

9º) The diagonals of a square are congruent.

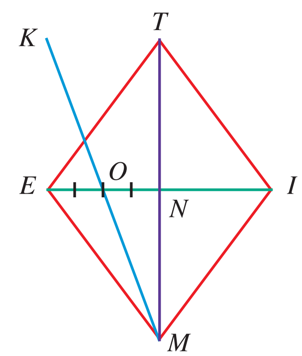
10º) A square is a rectangle having two consecutive congruent sides.

11º) A square admits four axes of symmetry.

For seeking

19 *TIME* is a rhombus of center *N*. *O* is the midpoint of [*EN*] and *K* the symmetric of *M* with respect to *O*.

Prove that *KENT* is a rectangle.



20 *ABC* is a triangle. [*AH*] is the altitude relative to [*BC*]. *M* is a point on (*AH*). *E* is the symmetric of *M* with respect to the midpoint *I* of [*AB*] and *F* the symmetric of *M* with respect to the midpoint *J* of [*AC*].

1º) Prove that *AEBM* and *AFCM* are two parallelograms.
Deduce that (*EB*) is perpendicular to (*BC*).

2º) Prove that *EFCB* is a rectangle.

21 *ABCD* is a rectangle of center *O*. *I* is the midpoint of [*OB*] and *O* is the symmetric of *A* with respect to *I*.

1º) What is the nature of quadrilateral *ABEO* ? Justify.
Deduce that : $BE = OC$.

2º) Prove that quadrilateral *OBEC* is a rhombus.

22 *O* is the center of a rhombus *ABCD*. *E*, *F*, *G* and *H* are respectively the feet of the perpendiculars drawn from *O* to the sides [*AB*], [*BC*], [*CD*] and [*AD*].

1º) What does [*BD*] represent for angle $\widehat{ABC}$? Justify your answer.

Deduce that $OE = OF$.

2º) Prove, in the same manner, that $OE = OH$.

3º) Prove that quadrilateral *EFGH* is a rectangle.

23 *ABC* is a triangle. *D* and *E* are respectively the symmetrics of *B* and *C* with respect to *A*.

1º) What is the nature of quadrilateral *BCDE* ? Justify your answer.

2º) How should the triangle *ABC* be so that *BCDE* would be :

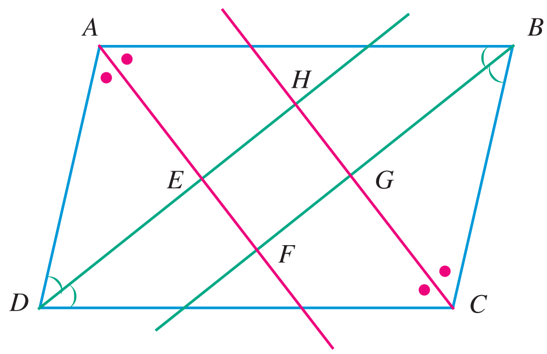
a) a rectangle ?

b) a rhombus ?

c) a square ?

24 The bisectors of the angles in a parallelogram *ABCD* intersect to form quadrilateral *EFGH*.

Prove that *EFGH* is a rectangle.



25 *ABCD* is a rectangle of center *I*.

1º) Place the point *L* so that *AIBL* would be a parallelogram.
Prove that *AIBL* is a rhombus.

2º) Place the point *N* so that *DICN* would be a parallelogram.
Prove that (*IN*) is perpendicular to (*DC*).

3º) Prove that the points *L*, *I* and *N* are collinear.

26 *DAB* is a right isosceles triangle at *D*.

1º) Place the point *C* so that *DABC* would be a parallelogram.

2º) The bisector of angle $\widehat{ADB}$ cuts [*AB*] at *E* and the bisector of angle $\widehat{DBC}$ cuts [*CD*] at *F*.
Prove that quadrilateral *DEBF* is a square.

TEST

1 *RON* is a right triangle at *R*. *A* and *I* are respectively the symmetrics of *O* and *N* with respect to *R*.

1º) Prove that *OIAN* is a rhombus. **(4 points)**

2º) *F* and *H* are respectively the midpoints of [*RO*] and [*RA*]. *C* and *D* are two points on [*IN*] such that $RC = RD = RF$.

What is the nature of quadrilateral *HCFD* ? Justify your answer. **(4 points)**

2 $\widehat{xOy}$ and $\widehat{yOz}$ are two adjacent supplementary angles. [*Ou*) is the bisector of $\widehat{xOy}$ and [*Ov*) is the bisector of $\widehat{yOz}$. *A* is a point on [*Oy*). *M* and *N* are respectively the feet of the perpendiculars drawn from *A* to [*Ou*) and [*Ov*).

1º) Prove that *AMON* is a rectangle. **(4 points)**

2º) Prove that : $(MN) // (xz)$. **(4 points)**

3 *ABC* and *DBC* are two isosceles triangles drawn on both sides of [*BC*]. *E* is the symmetric of *C* with respect to *A* and *F* is the symmetric of *C* with respect to *D*.

Prove that the points *E*, *B* and *F* are collinear. **(4 points)**

9 REMARKABLE IDENTITIES EXPANDING SIMPLIFYING

Objective

Perform operations using remarkable identities.

CHAPTER PLAN

COURSE

1. Reminder of certain rules of calculation

2. Order of calculation

3. Expanding (Development)

4. Simplifying (Reducing)

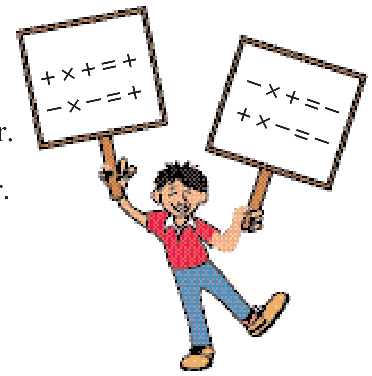
5. Development of certain products

EXERCISES AND PROBLEMS

TEST

COURSE

1 REMINDER OF CERTAIN RULES OF CALCULATION



Sign of product

⊙ The **product** of two numbers having the **same sign** is a **positive** number.

⊙ The **product** of two numbers with **opposite signs** is a **negative** number.

EXAMPLES

$(+4) \times (+7) = +28.$

$(-6) \times (-5) = +30.$

$(-3) \times (+9) = -27.$

$(+8) \times (-9) = -72.$

Removing parentheses

⊙ The following examples remind you of how to remove parentheses :

$A = 5 + (a - 3.5) = 5 + a - 3.5 = a + 1.5$;

$B = 4 + (-a - 2.7) = 4 - a - 2.7 = -a + 1.3$;

$C = 3 - (x - 7.1) = 3 - x + 7.1 = -x + 10.1$;

$D = -3 - (-x + 6.2) = -3 + x - 6.2 = x - 9.2$.

2 ORDER OF CALCULATION

⊙ Without parentheses, the operations are done in the following order :

1°) the powers.

2°) the multiplications and the divisions in the order of their appearance.

3°) the additions and the subtractions.

⊙ With parentheses, first remove them by calculating inside them, then calculate in the order indicated above.

EXAMPLES

$E = 3 \times (4)^2 - 2 \times 5 + 1$
$= 3 \times 16 - 2 \times 5 + 1$
$= 48 - 10 + 1$
$= 39$.

$F = -2 - (6 - 4 \times 3^2) - 4 \times 8 + 6$
$= -2 - 6 + 4 \times 3^2 - 4 \times 8 + 6$
$= -2 - 6 + 4 \times 9 - 4 \times 8 + 6$
$= -2 - 6 + 36 - 32 + 6$
$= +2$.

3 EXPANDING (DEVELOPMENT)

We remind you that : $\boldsymbol{k\,(a+b)=ka+kb}$.

To replace $k\,(a+b)$ by $ka+kb$ is to **develop** the product $k\,(a+b)$.

$\boldsymbol{ka}$ and $\boldsymbol{kb}$ are the **terms** of the expression «$ka+kb$».

EXAMPLES

$3\,(a+5)=3a+15$; $4\,(2a-3)=8a-12$; $-6\,(-a+2)=6a-12$.

4 REDUCING

In general, it is possible to **simplify** the expanded form of certain algebraic expressions. We say that we have **reduced** these algebraic expressions.

EXAMPLE

Reduce the algebraic expression. $E=2\,(a-7)-3\,(-a+1)-(4\,a-10)$.

Develop : $E=2a-14+3a-3-4a+10$.

Reduce the similar terms : $E=a-7$.

Application 1

Reduce each of the following expressions.
$A=2\,(a+2b-3)-4\,(-a+b-1)-(3a+b)$.
$B=3\,(x+9y-5)-2\,(4x-2y-7)$.
$C=5\,(a+3b-c)-6\,(2a+2b-3c)$.

Remarks

⊙ Written forms frequently used are such that :
$f(x)=4x^2+x-2$ and $A(m)=m^3-m+2$
This form is used to indicate the variable used in the given algebraic expression.
$f(x)$ is read «f of x»; x is the variable.
$A(m)$ is read «A of m»; m is the variable.

⊙ By substituting x by 2 in $f(x)$, we get :
$f(2)=4\,.\,2^2+2-2=16$.
$f(2)$, or 16, is called the numerical value of $f(x)$ for $x=2$.
Thus, the numerical value of $A(m)$ for $m=-1$ is :
$A(-1)\;=(-1)^3-(-1)+2$
$A(-1)\;=2.$

EXAMPLE

Consider the expression $E(x) = 3\,(x^2 - x) - x^2 + 2x - (3x^2 + 4)$.

It is written : $E(x) = 3x^2 - 3x - x^2 + 2x - 3x^2 - 4$
$= - x^2 - x - 4$.

Its numerical value for $x = 0$, is : $E(0) = - 0 - 0 - 4 = - 4$,

and its numerical value for $x = - 2$ is : $E(- 2) = - (- 2)^2 - (- 2) - 4$
$= - 4 + 2 - 4$
$= - 6$.

Application 2

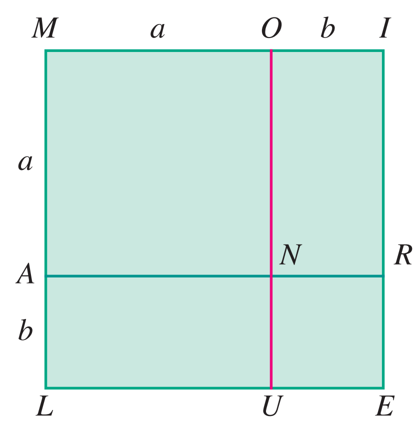
Given the algebraic expression : $E(x) = 5x\,(x - 2) - 3\,(x^2 + 1) - 3x + 7$.

1º) Develop and reduce $E(x)$.

2º) Calculate the numerical value of $E(x)$ for : $x = 0$; $x = \frac{2}{3}$; $x = \sqrt{2}$.

5 DEVELOPMENT OF CERTAIN PRODUCTS

Activity



The unit of length is the meter.

MIEL is a square. What is the measure of its side ?

Its area is equal to ......

Using a and b, calculate the following areas :

area of square *MONA* = ...

area of square *REUN* = ...

area of rectangle *NOIR* = ...

area of rectangle *LUNA* = ...

What can we say about the sum of these four areas ?

Complete then : $(a + b)^2$ = ... + ... + ... + ...

1°) Expanded form of *(a + b) (c + d)*

For any a, b, c and d, we have :

$$(a + b)(c + d) = ac + ad + bc + bd\ ,$$

$$(a - b)(c - d) = ac - ad - bc + bd\ .$$

(a + b) and ***(c + d)*** are the **factors** of the product $(a + b)(c + d)$.

(a – b) and ***(c – d)*** are the **factors** of the product $(a - b)(c - d)$.

EXAMPLE

Develop and reduce the following expressions :

1°) A = $(a + 2)(b + 3)$

A = $a \times b + 3 \times a + 2 \times b + 2 \times 3$
$= ab + 3a + 2b + 6$.

2°) B = $(2a - 3)(3a + 5)$.

B = $6a^2 + 10a - 9a - 15$
$= 6a^2 + a - 15$.

Application 3

Develop and reduce, if possible, the following algebraic expressions.

A = $(x + 2)\ (2x - 5)$; B = $(3x - 4)\left(-\,2x - \sqrt{3}\right)$;

C = $(x + 2y)\ (2x - y) + (3x - 2y)\ (-\,x + y)$.

2º) Remarkable Identities or products

⊙ By noticing that $(a + b)^2 = (a + b)\ (a + b)$, we get :

$$\begin{aligned}(a + b)^2 &= (a + b)\ (a + b)\\ &= a \times a + a \times b + b \times a + b \times b\\ &= a^2 + 2ab + b^2 \quad \text{(because } a \times b = b \times a = ab\text{).}\end{aligned}$$

⊙ We also calculate $(a - b)^2$ and $(a - b)\ (a + b)$:

$$\begin{aligned}(a - b)^2 &= (a - b)\ (a - b)\\ &= a \times a - a \times b - b \times a + b \times b\\ &= a^2 - 2ab + b^2 .\end{aligned}$$

$$\begin{aligned}(a - b)\ (a + b) &= a \times a + a \times b - b \times a - b \times b\\ &= a^2 - b^2 .\end{aligned}$$

$$\left(\underbrace{a}_{\substack{\text{1st}\\ \text{term}}} + \underbrace{b}_{\substack{\text{2nd}\\ \text{term}}}\right)^2 = \underbrace{a^2}_{\substack{\text{square of}\\ \text{the 1st}}} + \underbrace{2\,ab}_{\substack{\text{2 times the product}\\ \text{of 1st by the 2nd}}} + \underbrace{b^2}_{\substack{\text{square of}\\ \text{the 2nd}}}$$

$$\left(\underbrace{a}_{\substack{\text{1st}\\ \text{term}}} - \underbrace{b}_{\substack{\text{2nd}\\ \text{term}}}\right)^2 = \underbrace{a^2}_{\substack{\text{square of}\\ \text{the 1st}}} - \underbrace{2\,ab}_{\substack{\text{2 times the product}\\ \text{of 1st by the 2nd}}} + \underbrace{b^2}_{\substack{\text{square of}\\ \text{the 2nd}}}$$

$$\left(\underbrace{a}_{\substack{\text{1st}\\ \text{term}}} - \underbrace{b}\right)\left(\underbrace{a} + \underbrace{b}_{\substack{\text{2nd}\\ \text{term}}}\right) = \underbrace{a^2}_{\substack{\text{square of}\\ \text{the 1st}}} - \underbrace{b^2}_{\substack{\text{square of}\\ \text{the 2nd}}}$$

Examples

1º) Develop : **a)** $(x + 3)^2$; **b)** $(2y - 1)^2$; **c)** $(3x - 2)(3x + 2)$.

a) $(x + 3)^2 = x^2 + 2 \times x \times 3 + 3^2$
$= x^2 + 6x + 9.$

b) $(2y - 1)^2 = (2y)^2 - 2 \times (2y) \times 1 + 1^2$
$= 4y^2 - 4y + 1.$

c) $(3x - 2)(3x + 2) = (3x)^2 - (2)^2$
$= 9x^2 - 4.$

2º) Calculate quickly : **a)** 41^2 ; **b)** 49^2 ; **c)** 59×61.

a) $41^2 = (40 + 1)^2$
$= 40^2 + 2 \times 40 \times 1 + 1^2$
$= 1600 + 80 + 1$
$= 1681.$

b) $49^2 = (50 - 1)^2$
$= 50^2 - 2 \times 50 \times 1 + 1^2$
$= 2500 - 100 + 1$
$= 2401.$

c) $59 \times 61 = (60 - 1)(60 + 1)$
$= 60^2 - 1$
$= 3600 - 1$
$= 3599.$

Application 4

1º) Develop each of the following expressions :

a) $(4x + 3)^2$; **b)** $\left(x - \frac{3}{2}\right)^2$; **c)** $\left(4x - \frac{3}{2}\right)\left(4x + \frac{3}{2}\right)$.

2º) Calculate using remarkable identities :

a) 51^2 ; **b)** 39^2 ; **c)** 899×901 .

EXERCISES AND PROBLEMS

Test your knowledge

1 Develop, reduce, then calculate the numerical value of each of the following expressions, for $a = 3$ and $b = -4$.

1º) $E = 7 - (4 - 3a) - (2a - b) + (b - 2)$; **2º)** $F = -3 + (-14 + b) - (a - 13 + b)$.

2 Given the following algebraic expressions : $A(x) = x^2 - 3x + 2$ and $B(x) = 3x^2 - x - 2$.

Calculate : $A(1)$; $A(2)$; $A(0)$; $B(1)$; $B\left(-\frac{2}{3}\right)$ and $B\left(\sqrt{3}\right)$.

3 Develop and reduce :

1º) $a(a + b + 2) - 2a(a + 1)$ **2º)** $3(x^2 - 2x - 3) - 3x(x + 2)$

3º) $x(x - 5) + 5(x - 5)$ **4º)** $2 - y(2y - 1) + 3(y^2 + 1)$.

4 Develop and reduce each of the following expressions.

1º) $(a + 2)(b + 3)$. **2º)** $(x - 6)(y - 4)$.**3º)** $(a - 3)(b + 5)$.

4º) $(2x - 3)(2x + 3)$. **5º)** $(3a - 2)(2a + 3)$. **6º)** $(3x + 1)(3y - 1)$.

5 Develop.

1º) $(x + 2)^2$. **2º)** $(2x + 3)^2$. **3º)** $(x - \sqrt{6})^2$.

4º) $(3x - 1)^2$. **5º)** $(2x + y)^2$. **6º)** $(2x - y)^2$.

7º) $(3x + 2y)^2$. **8º)** $(3x - 2y)^2$. **9º)** $(-2x + y)^2$.

10º) $(-2x - y)^2$. **11º)** $\left(x + \frac{1}{2}\right)^2$. **12º)** $\left(2x - \frac{1}{4}\right)^2$.

6 Develop.

1º) $\left(x+\dfrac{2}{3}\right)\left(x-\dfrac{2}{3}\right)$.

2º) $\left(2x-\dfrac{1}{3}\right)\left(2x+\dfrac{1}{3}\right)$.

3º) $(4x-3y)\,(4x+3y)$.

4º) $\left(x+\dfrac{y}{2}\right)\left(x-\dfrac{y}{2}\right)$.

5º) $\left(a+\dfrac{2b}{3}\right)\left(-a+\dfrac{2b}{3}\right)$.

6º) $\left(2a-\dfrac{b}{5}\right)\left(2a+\dfrac{b}{5}\right)$.

7 **1º)** Using the fact that : $31 = 30 + 1$ and $29 = 30 - 1$, calculate :

a) 31^2 ; **b)** 29^2 ; **c)** 31×29 .

2º) Use remarkable identities to calculate.

a) 49×51. **c)** 37×43. **e)** 21^2 . **g)** 68^2 .

b) 38×42. **d)** 33×27. **f)** 39^2 . **h)** 52^2 .

8 Develop and reduce.

1º) $\left(\dfrac{2}{3}x-\dfrac{3}{5}\right)\left(\dfrac{5}{3}y-3\right)$.

2º) $\left(\dfrac{x}{4}+\dfrac{2}{3}\right)\left(\dfrac{x}{4}-\dfrac{2}{3}\right)$.

3º) $\left(\dfrac{x}{3}-\dfrac{3}{2}\right)^2$.

4º) $\left(\dfrac{2x}{3}-\dfrac{1}{4}\right)^2$.

9 Develop and reduce.

1º) $A = (2x+1)^2 + (x-1)\,(3x-1)$.

2º) $B = (x-3)^2 - 3x\,(2x+1)$.

3º) $C = (x-2)\,(x+2) - (x-3)^2$.

4º) $D = (x-2y)^2 + (2x+y)^2$.

5º) $E = \left(\dfrac{x}{2}+y\right)^2 + \left(\dfrac{x}{2}-y\right)^2$.

6º) $F = (3x-y)^2 - (3x+y)^2$.

10 Develop and reduce.

1º) $A = 3\,(x-5)^2 - 2\,(x-6)^2$.

2º) $B = 4\,(-x+2)^2 - 3\,(-x+8)\,(-2x+2)$.

3º) $C = 4\,(a-2)^2 - 3a\,(1-2a) - 2a + 1$.

4º) $D = (2x-3)^2 + (2x-3)\,(2x+3)$.

5º) $E = (3x-2)^2 - (-3x+2)^2$.

6º) $F = (5x-1)^2 - (5x+1)^2$.

11 Complete.

1º) $(... + 4)^2 = x^2 + ... + 16.$ **2º)** $(x - ...)^2 = x^2 - ... + 9.$

3º) $(... + 2)^2 = 9a^2 + ... + ...$ **4º)** $\left(... + \frac{1}{2}\right)^2 = x^2 + ... + ...$

12 Given the sum of two terms of the square of a binomial; find the third term and write the square of the binomial.

1º) $a^2 - 2ab.$ **2º)** $y^2 + 8y.$ **3º)** $9x^2 - 6x.$ **4º)** $4a^2 + 20a.$

For seeking

13 Develop.

1º) $(m^2 - 4)^2.$ **2º)** $(2m^2 - 3a^2)^2.$

3º) $(3b^2 + 4a^3)^2.$ **4º)** $(\sqrt{2}x + y)^2.$

5º) $\left(m^3 + \frac{1}{2}\right)\left(m^3 - \frac{1}{2}\right).$ **6º)** $(3c^4 + 2c^2)(3c^4 - 2c^2).$

7º) $(-5ax^2 + 2by)(5ax^2 + 2by).$ **8º)** $\left(\frac{4}{5}ax^2 + \frac{3}{2}y^3\right)\left(\frac{4}{5}ax^2 - \frac{3}{2}y^3\right).$

9º) $(2x - \sqrt{3})(2x + \sqrt{3}).$ **10º)** $(\sqrt{7}x + \sqrt{5})(\sqrt{7}x - \sqrt{5}).$

11º) $(x\sqrt{2} - y)(y\sqrt{2} - x).$ **12º)** $\left(\frac{3}{7}a^3 + \frac{7}{6}b^3\right)^2.$

14 Given the following algebraic expression : $A(x) = (2x + 3)^2 - (2x - 3)^2 - 24x + 3.$

1º) Calculate : $A(0)$; $A(1)$; $A(-1)$.

2º) Do we have the same value of $A(x)$ for any value of x ? Explain.

15 Develop and reduce the expression $E = (2x + y)^2 - (2x - y)^2 + 4$; then calculate the numerical value of E for $xy = -1$.

16 Develop $\left(x + \frac{1}{x}\right)^2$; then calculate $x^2 + \frac{1}{x^2}$, if $x + \frac{1}{x} = 3.$

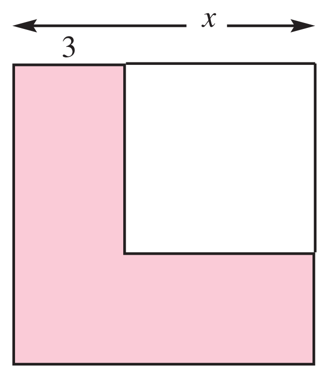
17 **1º)** Given : $a = x + y$.

Develop $(a + 2)(a - 2)$; Deduce the expanded form of $(x + y + 2)(x + y - 2)$.

2º) By using a remarkable identity, develop :

$(a - b - 3)(a + b + 3)$.

18 A square has a side x expressed in cm ($x > 3$). Decrease the side by 3 cm.



1º) Among the following three algebraic expressions, which one shows the decrease of the area of the square :

$(x + 3)^2 - x^2$; $x^2 - (x - 3)^2$; $(x - 3)^2 - x^2$.

2º) **a)** Show that this decrease is equal to : $6x - 9$.

b) Calculate its numerical value when the side of the square is equal to 6 cm.

19 Consider the following expressions : $(3 + x)^2$; $(-3 + x)^2$; $(x + 3)(3 - x)$; $(-x - 3)^2$; $(3 - x)^2$; $(3 + x)(x - 3)$; $(-x + 3)^2$; $(-x - 3)(-x + 3)$.

Without developing them, find those that are equal to :

1º) $(x + 3)^2$. **2º)** $(x - 3)^2$. **3º)** $(x + 3)(x - 3)$.

20 **1º)** By noticing that $3 \times 5 = (4 - 1) \times (4 + 1)$, calculate : $4^2 - 3 \times 5$.

Also calculate : $10^2 - 9 \times 11$; $23^2 - 22 \times 24$.

2º) Three consecutive numbers can be denoted by : $n - 1$; n ; $n + 1$.

Develop and reduce : $n^2 - (n - 1)(n + 1)$, where $n > 1$.

Deduce the value of : $5346819^2 - 5346818 \times 5346820$.

21 Three consecutive numbers can be denoted by : n , $n + 1$ and $n + 2$.

1º) Develop and reduce : $(n + 1)^2 - n(n + 2)$.

2º) Deduce the value of $372517^2 - 372516 + 372518$.

TEST

1 Answer by true or false. **(4.5 points)**

1º) Whatever the numbers a, b and c, we get the following :

$2a - 3\,(a - 2b) + 2\,(a - 3b) - a + 2 = 2.$

2º) If $E(x) = 4\,(x - 3) - 7\,(x - 3)^2$, then $E(3) = 0$.

3º) For any numbers x and y, we get : $(3x + 6y)^2 = 9x^2 + 36y^2$.

2 Relate the equal expressions. **(4.5 points)**

$a^2 - 9$	•	•	$(5x + 2)^2$
$\frac{x^2}{4} + x + 1$	•	•	$\left(\frac{x}{2} - 1\right)\left(\frac{x}{2} + 1\right)$
$3a^2 - 1$	•	•	$(a - 3)\,(a + 3)$
$25x^2 + 20\,x + 4$	•	•	$\left(\frac{x}{2} + 1\right)^2$
$\frac{x^2}{4} - 1$	•	•	$\left(\frac{a}{3} - 1\right)^2$
$\frac{a^2}{9} - \frac{2a}{3} + 1$	•	•	$(\sqrt{3}\,a + 1)\,(\sqrt{3}\,a - 1)$

3 A square has a side x expressed in cm; we increase its side by 2 cm.

Using x, express the increase of the area, then calculate this increase when the side of the square is equal to 4 cm. **(4 points)**

4 Calculate the numerical value of :

$A = (x + 2)^2 - 3x\,(y + 2) - x^2 - 2y$ and $B = (3x - y)^2 - (3x + y)^2 - 4\,(x + y)^2$,

knowing that $xy = -5$ and $x + y = 4$. **(6 points)**

10

FACTORIZATION

Objectives

1. To find a common factor in an algebraic expression.

2. To use the remarkable identities in factorizing an algebraic expression.

CHAPTER PLAN

COURSE

1. Factorization

2. Remarkable identities and factorization

EXERCISES AND PROBLEMS

TEST

1 FACTORIZATION

Activity

You know that :

$$ka + kb = k\,(a + b) \quad ; \quad ka - kb = k\,(a - b)$$
$$ka + kb + kc = k\,(a + b + c) \quad ; \quad ka - kb - kc = k\,(a - b - c)$$

Factorize :

1°) $3x - 9$; **2°)** $5x + 10y$; **3°)** $x + xy$

4°) $2x + 2y - 2$; **5°)** $xy + x^2 + x$; **6°)** $-5xy + 10x + 5x$.

2 REMARKABLE IDENTITIES AND FACTORIZATION

The following chart shows the factorization of some algebraic expressions.

Developed expression	Factorized expression
$a^2 + 2ab + b^2$	$(a + b)^2$
$a^2 - 2ab + b^2$	$(a - b)^2$
$a^2 - b^2$	$(a - b)\,(a + b)$

Examples

Factorize the following algebraic expressions.

1°) $A = \underline{5}x + \underline{5}y - \underline{5}z.$

The common factor is 5.

Thus : $A = 5\ (x + y - z).$

2°) $B = 3\underline{x} + 2\underline{x}y - \underline{x}z.$

The common factor is x.

Thus : $B = x\ (3 + 2y - z).$

3°) $C = a^2b + a^3c + a^2d.$

a^2 appears in the three terms; a^2 is a common factor. Thus :

$C = a^2\ (b + ac + d).$

4°) $D = 2\ \underline{(x - 1)} + 3a\ \underline{(x - 1)}.$

The common factor is $(x - 1)$.

$D = (x - 1)\ (2 + 3a).$

5°) $E = (x + 1)\ (x - 2) + 6(x - 2)^2\ .$

$E = (x + 1)\ \underline{(x - 2)} + 6\underline{(x - 2)}\ (x - 2).$

$= (x - 2)\ [(x + 1) + 6(x - 2)]$

$= (x - 2)\ (7x - 11).$

6°) $F = (x - 5)\ (2x + 3) + x^2 - 25.$

$F = \underline{(x - 5)}\ (2x + 3) + \underline{(x - 5)}\ (x + 5)$

$= (x - 5)\left[(2x + 3) + (x + 5)\right]$

$= (x - 5)\ (3x + 8).$

7°) $G = (3x + 2)^2 - (x + 5)^2\ .$

G is the difference of two squares.

$G = \left[(3x+2)+(x+5)\right]\left[(3x+2)-(x+5)\right]$

$= (4x + 7)\ (2x - 3).$

8°) $H = 9y^2 + 12y + 4 = (3y + 2)^2\ .$

9°) $I\ = 4a^2 - 4ab + b^2 = (2a - b)^2\ .$

10°) $J\ = \underline{a}x + \underline{x} + \underline{a}y + \underline{y} = x\ \underline{(a + 1)} + y\underline{(a + 1)}$

$= (a + 1)\ (x + y).$

EXERCISES AND PROBLEMS

Test your knowledge

1 Factorize each of the following expressions.

1º) $A = 42 - 6x$. **2º)** $B = 35a - 7b$. **3º)** $C = 48b + 8$.

4º) $D = 8a + 24b - 16$. **5º)** $E = 15a - 45b + 30$. **6º)** $F = 20.8x - 5.2y + 10.4$.

2 The common factor is a monomial. Factorize.

1º) $A = 2x - ax$. **2º)** $B = a^3 + 5a$.

3º) $C = 2a^3 + 3a^2$. **4º)** $D = 2ax + 3ab + a^2c$.

5º) $E = 4x^2 - 5yx + 4x$. **6º)** $F = 5a^2 + 3a^2b - 2a^2c$.

3 Same exercise.

1º) $A = 16x^3 - 32x^2$. **2º)** $B = 11x^2 - 121x$.

3º) $C = 21a^2y - 14ay^2$. **4º)** $D = 24a^3 + 8a^2 + 16a$.

5º) $E = 15b^3 - 30b^2 + 15b$. **6º)** $F = 6x^2y - 18xy^2 + 6xy$.

4 The common factor is a binomial. Factorize.

1º) $A = a(2a - 3) - 3(2a - 3)$. **2º)** $B = y(3x + 4) + 6(3x + 4)$.

3º) $C = 6x(2a + 5) - 5y(2a + 5)$. **4º)** $D= 6a(3a - 2) + 12b(3a - 2)$.

5º) $E = 4x(3y + 1) - 8y(3y + 1)$. **6º)** $F = 12x(4a - 1) - 4(4a - 1)$.

5 Same exercise.

1º) $A = (a + 4)(2x - 3) + (a + 4)(3x + 1)$. **2º)** $B = (2a - 3)(4b - 1) - 2b(2a - 3)$.

3º) $C = (2a - 1)(4a + 9) + (4a + 9)(3a - 2)$. **4º)** $D = 2x(x + 3) - 5y(x + 3) + 3z(x + 3)$.

5º) $E = 6(2a - 3)(a + 7) + 18(2a - 3)(3a + 2)$. **6º)** $F = (2x - 5)(3x + 2) - (4x - 10)(5x + 4)$.

6 Factorize the following using the difference of two squares.

1º) $4x^2 - 9$. **2º)** $(x + 4)^2 - 36$. **3º)** $y^2 - (2x + 3)^2$.

4º) $(x + 5)^2 - (9 - 2x)^2$. **5º)** $(2y - 3)^2 - (y - 5)^2$. **6º)** $(5y + 3)^2 - (7 + 2y)^2$.

7º) $(x + 2)^2 - 16(3x - 1)^2$. **8º)** $9(z + 4)^2 - 4z^2$. **9º)** $100y^2 - \frac{4}{9}y^4$.

7 Same exercise.

1º) $A = 121x^2 - 81a^2y^2$. **2º)** $B = 75a^2b^2 - 27x^2y^2$.

3º) $C = (a - 2)^2 - b^2$. **4º)** $D = (2a + 1)^2 - 16b^2$.

5º) $E = \frac{9x^2}{25} - (y - 2)^2$. **6º)** $F = 4(a - 3)^2 - (b + 2)^2$.

7º) $G = (2a - 5)^2 - 9(3a + 2)^2$. **8º)** $H = \frac{(x - 2y)^2}{100} - \frac{(2x - y)^2}{36}$.

8 The following expressions are perfect squares. Factorize.

1º) $A = x^2 + 8x + 16$. **2º)** $B = 4x^2 - 12x + 9$. **3º)** $C = 121a^2 - 22a + 1$.

4º) $D = 9m^2 - 12m + 4$. **5º)** $E = 9x^2 + 3xy + \frac{y^2}{4}$. **6º)** $F = \frac{4x^2}{9} + \frac{4x}{3} + 1$.

9 The common factor appears after a change of signs. Factorize.

1º) $A = (3a - x)(b + 2x) - (x - 3a)(b - 2x)$.

2º) $B = (x - y)(2x - y + z) + (y - x)(x - y + z)$.

3º) $C = (a - 2b)(x - y) + (2b - a)$.

4º) $D = (x - y)(z - x - y) - (x + y - z)(x + 2y)$.

5º) $E = (a - 2b)(x - y) + (2b - a) - (a - 2b)$.

10 The common factor appears after the factorization of a remarkable identity. Factorize.

1º) $A = x^2 + 2x + 1 - (x + 1)(3x - 2)$.

2º) $B = x^2 - 16 + x(x - 4)$.

3º) $C = (y + 5)(2y - 1) - y^2 + 25$.

4º) $D = \frac{9}{4}x^2 - x + \frac{1}{9} + \left(\frac{3x}{2} - \frac{1}{3}\right)\left(\frac{x}{3} + 1\right)$.

11 Group and factorize.

1º) $A = ax + by - ay - bx$.

2º) $B = a^2x - b^2y - a^2y + b^2x$.

3º) $C = 5bx - ay + by - 5ax$.

4º) $D = by - bx + 3ax - 3ay$.

5º) $E = ax^2 + yz - axy - xz$.

6º) $F = x^2 - y^2 - 7x + 7y$.

7º) $G = 9x^2 + 3xy - 3ax - ay$.

8º) $H = 6x^2 - 6y + ay - ax^2$.

9º) $I = yz - x^2 + xz - xy$.

10º) $J = a^2 + 2ab + b^2 - 16$.

12 Answer by true or false.

1º) x^2 is a common factor among the terms of the following sum : $3x^2 + 2x^2y + x^3z$.

2º) A factorized form of $4a^2 - 3b^2$ is $(2a - 3b)(2a + 3b)$.

3º) A factorized form of $9a^2 - 6a + 1$ is $(3a - 1)^2$.

4º) In the expression : $2(a - 1) - 3x(a - 1)^2$, $(a - 1)$ is a common factor.

5º) A factorized form of $ax - 2bx + x$ is $x(a - 2b)$.

6º) A factorized form of $x^2 - 100$ is $(x - 10)^2$.

For seeking

13 Factorize the following expressions.

1º) $A = 4x(3x - 1) - (x + 2)(3x - 1) + 3x - 1.$

2º) $B = (2x + 1)(4x + 3) - 5x(4x + 3) + (x - 1)(4x + 3).$

3º) $C = (x - 2)(5x - 1) + (x - 3)(5x - 1) - (3x + 2)(5x - 1).$

4º) $D = (3a + 1)(a + 1) + a^2 - 1.$

5º) $E = 4y^2 - 9 + (2y + 3)(y - 5).$

6º) $F = (x - 3)(2x + 7) + (2x - 6)(3x - 1) - (9 - 3x)(x + 1).$

7º) $G = (4x - 3)(-x + 5) + (x - 1)(x - 5) + (2x - 5)(-x + 5).$

8º) $H = 6(x^2 - 16) - (3x + 1)(x - 4) + (8 - 2x)(x + 2).$

9º) $I = (x + 7)(3x + 4) + (9x^2 + 24x + 16).$

10º) $J = 3x^2 - 12 + (x - 4)(2 - x) - (x^2 - 4x + 4).$

11º) $K = (6x^2 - 12x + 6) + (3x^2 - 3) - (x - 1)(2x + 1).$

12º) $L = 4x^2 - 4x + 1 - (1 - 2x)(3x + 5) - 12x^2 + 3.$

13º) $M = (3a - 2)(2a + 1) - (3a - 2)^2.$

14º) $N = 2x(x^2 - 1) - x(x + 1).$

14 Factorize the following expressions.

1º) $A = a^3 + a^2 + a + 1.$

2º) $B = xy - 3x - 2y + 6.$

3º) $C = 2 - b - 2a + ab.$

4º) $D = 10xy - 2 + 4x - 5y.$

5º) $E = -x^2 - y^2 + a^2 + b^2 + 2xy - 2ab.$

6º) $F = 25(3x - y)^2 - 16(5x + 3y)^2.$

7º) $G = \left(\frac{x}{4} - \frac{1}{3}\right)^2 - \left(\frac{5x}{4} - \frac{2}{3}\right)^2.$

8º) $H = (2a - 3)^2 - 2(2a - 3) + 1.$

9º) $I = (3x+2)^2 + 2(3x+2)(x-1) + (x-1)^2.$

10º) $J = (a+4)^2 - 2(a+4)(2a+1) + (2a+1)^2.$

11º) $K = 4x^2 - 4xy + y^2 - 9x^2y^2.$

12º) $L = 5(x^2-4) - x^2 + 4x - 4 + (6-3x)(x+3).$

13º) $M = 3(x - 1)^2 - x^2 + 1 + (x - 1)(x + 2).$

14º) $N = 25x^2 + (5x - 3)(2x + 7) - 9.$

15º) $O = (3x^2 - 25)^2 - 4x^4.$

16º) $P = (4a^2 + 1)^2 - (5a^2 - 2)^2.$

17º) $Q = a^2 - ab - b - 1.$

18º) $R = x^2 - 4x + (x - 4)^2.$

19º) $S = 3(5x - 1)^2 - 3(x + 2)^2.$

20º) $T = \left(\frac{x}{2} - 1\right)^2 - \frac{25}{16}.$

21º) $U = \frac{x^2}{4} - \frac{xy}{2} + \frac{y^2}{4}.$

22º) $V = x^2 - 7x + 6.$

15 Consider the expression $A = 6x^3 - 24x^2 + 24x$.

1º) Calculate A for $x = -2$; $x = 2$; $x = 0$.

2º) Factorize A, then calculate the numerical value of A for $x = -2$; $x = 2$; $x = 0$.

16 Consider the expression : $A(x) = 16 - (2x + 3)^2$.

1º) Develop $A(x)$ and reduce the obtained expression.

2º) Factorize $A(x)$.

3º) Use the developed and the factored form to calculate the numerical value of $A(x)$ for :

a) $x = 0$; **b)** $x = \frac{1}{2}$; **c)** $x = -2$; **d)** $x = -\frac{7}{2}$.

17 Given the expression : $E(x) = (5x - 1)^2 - (7x + 2)(5x - 1)$.

1º) Develop and reduce $E(x)$.

2º) Factorize $E(x)$.

3º) Calculate $E\left(\frac{1}{5}\right)$, $E\left(-\frac{3}{2}\right)$ and $E(0)$.

18 Consider the expression : $P(a) = (3a - 2)(4a - 3) - 9a^2 + 4$.

1º) Expand, reduce and order $P(a)$.

2º) Factorize $P(a)$.

3º) Calculate the numerical value of $P(a)$, in each of the following cases :

a) $a = \frac{2}{3}$.

b) $a = -\frac{1}{2}$.

c) $a = 0$.

19 Given the expression : $A(x) = (6x - 4)(x + 1) - (15x - 10)(4 - x) + 9x^2 - 12x + 4$.

1º) Expand, reduce and order $A(x)$.

2º) Factorize $A(x)$.

3º) Calculate the numerical value of $A(x)$ in each of the following cases :

a) $x = 0$.

b) $x = \frac{2}{3}$.

4º) Which one is the form (developed or factorized) that requires the minimum calculations to calculate $A(x)$ for each of the already given values of x ?

20 Given $E = x^2 + 6x + 5$.

1º) Develop $(x + 3)^2$.

2º) Calculate a for having $E = (x + 3)^2 - a$.

3º) Factorize E .

21 Given $F = x^2 - 4x - 5$.

1º) Develop $(x - 2)^2$.

2º) Calculate b for having $F = (x - 2)^2 - b$.

3º) Factorize F .

22 Given $G = x^2 + x - 2$.

1º) By adding and by subtracting $\frac{1}{4}$ to G , show that $G = \left(x + \frac{1}{2}\right)^2 - c$, where c is a constant to be determined.

2º) Factorize G .

23 The area of a rectangle is given by the following expression : $10ab + 5a + 6b + 3$, where a and b are two positive numbers.

Which expressions can show the dimensions of this rectangle ?

24 The area of a square is given by the following expression : $9x^2 + 42x + 49$, where x is a positive number.
Which expression shows the measure of the side of this square ?

TEST

1 Answer by true or false. **(4 points)**

1º) $(y-5)^2$ is a factorization of $y^2 - 25$.

2º) The expression $A = 5 \times x + 10 \times y$ is factored.

3º) $2x^2$ is a common factor of the terms of this sum :

$4x^2y + 8x^2y^2 - 6x^3z$.

4º) In the expression : $(2a-4)(a+3) - (a^2-4)$, $(a-2)$ is a common factor.

2 Factorize. **(3 points)**

1º) $ax + ay$; **2º)** $x^2 + xy$; **3º)** $ax + ay + x^2 + xy$.

3 Factorize. **(6 points)**

1º) $A = (5x-3)(x-6) - (x+4)(6-10x)$.

2º) $B = (x-5)(4-x) + \left(2 - \frac{x}{2}\right)(x+8)$.

3º) $C = \frac{1}{100}(x-1)^2 - 1$.

4º) $D = 4(x-2)^2 - 9(2x+1)^2$.

4 Given the expression : $A(x) = (2x-3)(x+2) - 12x^2 + 27 + (2x-3)^2$.

1º) Develop, reduce and order $A(x)$, then calculate :

$A(0)$; $A\left(\frac{3}{2}\right)$; $A\left(-\frac{10}{3}\right)$. **(2 points)**

2º) Write $A(x)$ as product of factors, then calculate :

$A(0)$; $A\left(-\frac{10}{3}\right)$; $A(5)$. **(2 points)**

5 Given $A = x^2 - 7x + 6$.

1º) Develop $\left(x - \frac{7}{2}\right)^2$.

2º) Calculate the constant k so that $A = \left(x - \frac{7}{2}\right)^2 + k$.

3º) Factorize A . **(3 points)**

11

TRAPEZOID MIDSEGMENT THEOREM

Objectives

1. To know the properties of a trapezoid.
2. To know and use the midsegment theorem in a triangle and in a trapezoid.
3. To know and to use the properties of an isosceles trapezoid.

CHAPTER PLAN

COURSE

1. Definition of a trapezoid
2. Special trapezoids
3. Properties of an isosceles trapezoid
4. How to prove that a trapezoid is isosceles
5. Midsegment theorem
6. Properties

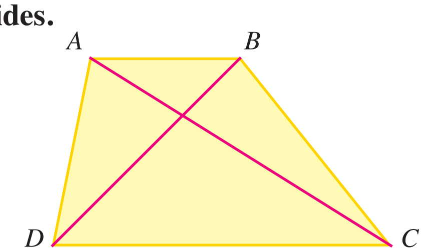
EXERCISES AND PROBLEMS

TEST

COURSE

1 DEFINITION OF A TRAPEZOID

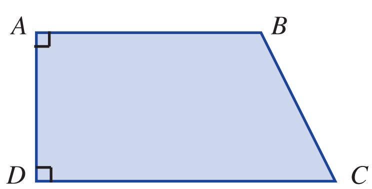
A trapezoid is a quadrilateral having exactly two parallel sides.



The adjacent figure shows a trapezoid *ABCD*.

- The sides [*AB*] and [*CD*] are parallel; **[*CD*]** is the **big base**, **[*AB*]** is the **small base**.
- [*AD*] and [*BC*] are the nonparallel sides. They are called the legs of the trapezoid.
- [*AC*] and [*BD*] are the diagonals.
- The distance between the bases is the **height** of the trapezoid.

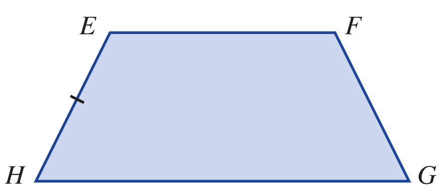
2 SPECIAL TRAPEZOIDS



Right trapezoid

$\widehat{BAD} = \widehat{ADC} = 90°$

[*AD*] is the height of the trapezoid

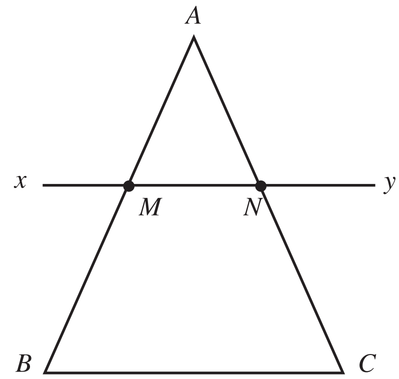


Isosceles trapezoid

The nonparallel sides are congruent : *EH* = *FG*

3 PROPERTIES OF AN ISOSCELES TRAPEZOID

Activity



Given an isosceles triangle ABC of vertex A. (xy) is a line parallel to (BC) ; (xy) cuts $[AB]$ at M and $[AC]$ at N.

1º) Prove that triangle AMN is isosceles.
Deduce that $MB = NC$.

2º) Are the opposite sides of quadrilateral $MNCB$ parallel ?

Name the equal angles in this quadrilateral. Justify your answer.

3º) Prove that the two triangles MBC and NCB are congruent.

Deduce that $MC = NB$.

4º) $[MN]$ and $[BC]$ have the same perpendicular bisector; justify. Construct this perpendicular bisector. Is it an axis of symmetry in the figure ?

Properties

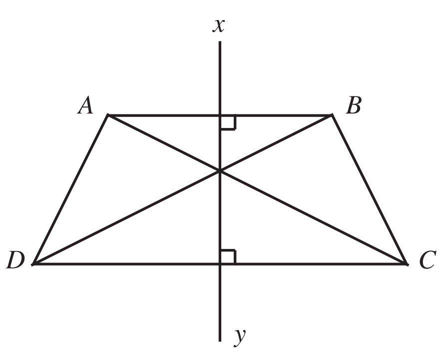
In an **isosceles trapezoid** :

1º) The **angles adjacent to the big base are equal.**
It is **the same** for **the angles adjacent to the small base**.

2º) The **diagonals** are **congruent**.

3º) The **bases have the same perpendicular bisector**. This perpendicular bisector is the axis of symmetry of the trapezoid.

EXAMPLE



The adjacent trapezoid $ABCD$ is isosceles.
Therefore :

⊙ $AD = BC$.

⊙ $\widehat{ADC} = \widehat{BCD}$ and $\widehat{DAB} = \widehat{CBA}$.

⊙ $AC = BD$.

⊙ (xy) is the perpendicular bisector of both bases $[AB]$ and $[DC]$. (xy) is the axis of symmetry of the trapezoid.

Application 1

ABCD is an isosceles trapezoid with [*AB*] and [*CD*] as bases and such that $\widehat{BAD} = 75°$.

Calculate the other angles of this trapezoid.

4 HOW TO PROVE THAT A TRAPEZOID IS ISOSCELES ?

To prove that a **trapezoid is isosceles**, prove one of the following **properties** :

1º) The angles adjacent to one of the bases are equal.

2º) The two nonparallel sides are congruent.

3º) The diagonals are congruent.

4º) The bases have the same perpendicular bisector.

Application 2

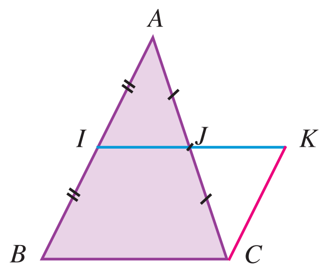
ABCD is a rectangle. *M* and *N* are two points on [*AB*] such that *AM* = *BN*.

Prove that the points *M*, *N*, *C* and *D* are the vertices of an isosceles trapezoid.

5 MIDSEGMENT THEOREM

Activity

ABC is a triangle. *I* is the midpoint of [*AB*], *J* is the midpoint of [*AC*] and *K* is the symmetric of *I* with respect to *J*.



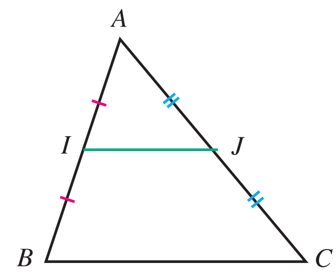
1º) Prove that quadrilateral *AKCI* is a parallelogram.
Deduce that *CK* = *BI*.

2º) Prove that *CKIB* is a parallelogram.
Deduce that : $(IJ) \,//\, (BC)$ and $IJ = \frac{1}{2} BC$.

Theorem

The segment joining the midpoints of two sides in a triangle is parallel to the third side and its length is half the length of the third side.

EXAMPLE



In the adjacent triangle ABC , I is the midpoint of $[AB]$ and J the midpoint of $[AC]$, therefore :

$(IJ) \mathbin{//} (BC)$ and $IJ = \dfrac{BC}{2}$.

Application 3

ABC is a right triangle at A. M, I and J are respectively the midpoints of $[BC]$, $[AC]$ and $[AB]$.

Use the midsegment theorem to prove that AMC and AMB are two isosceles triangles.

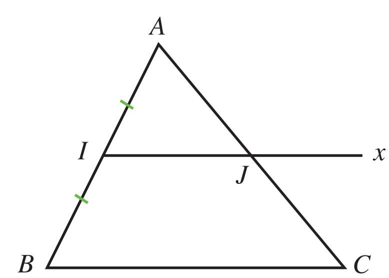
Deduce that $AM = \dfrac{1}{2} BC$.

6 PROPERTIES

Property 1

In a triangle, **the line drawn from the midpoint of a side parallel to another, passes through the midpoint of the third side.**

EXAMPLE



In the adjacent triangle ABC , I is the midpoint of $[AB]$; (Ix) is parallel to (BC).

Therefore, (Ix) passes through the midpoint J of $[AC]$.

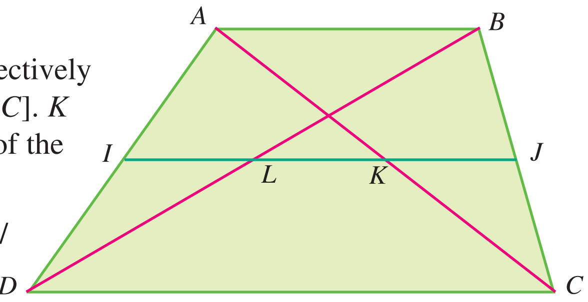
Application 4

$ABCD$ is a parallelogram of center O. The parallel drawn from O to (AB) cuts $[AD]$ at I and $[BC]$ at J.

Prove that I is the midpoint of $[AD]$ and that J is the midpoint of $[BC]$.

Property

2



ABCD is a trapezoid. *I* and *J* are respectively the midpoints of the legs [*AD*] and [*BC*]. *K* and *L* are respectively the midpoints of the diagonals [*AC*] and [*BD*]. Therefore :

1º) *I, L, K* and *J* are collinear. $(IJ) \parallel (AB) \parallel (CD)$.

([*IJ*] is called the midsegment of the trapezoid).

2º) $IJ = \dfrac{AB + CD}{2}$ (Half the sum of the bases of the trapezoid).

3º) $LK = \dfrac{CD - AB}{2}$ (Half the difference of the bases of the trapezoid).

Proof

1º) In triangle *ABD*, *I* is the midpoint of [*AD*] and *L* is the midpoint of [*BD*].

Therefore (*IL*) is parallel to (*AB*) and $IL = \dfrac{1}{2} AB$.

Moreover :

In triangle *DBC*, $(LJ) \parallel (DC)$ and $LJ = \dfrac{1}{2} CD$;

In triangle *ABC*, $(JK) \parallel (AB)$ and $JK = \dfrac{1}{2} AB$.

The lines (*IL*) and (*LJ*) pass through *L* and are parallel to (*AB*) and (*CD*); therefore, they are confounded.

Consequently *I, L* and *J* are collinear.

The lines (*LJ*) and (*JK*) pass through *J* and are parallel to (*CD*) and (*AB*); therefore, they are confounded.

Consequently *L, J* and *K* are collinear.

We deduce that the points *I, L, K* and *J* are collinear.

2º) $IJ = IL + LJ = \dfrac{1}{2} AB + \dfrac{1}{2} CD = \dfrac{AB + CD}{2}$.

3º) $LK = LJ - JK = \dfrac{1}{2} CD - \dfrac{1}{2} AB = \dfrac{CD - AB}{2}$.

Application 5

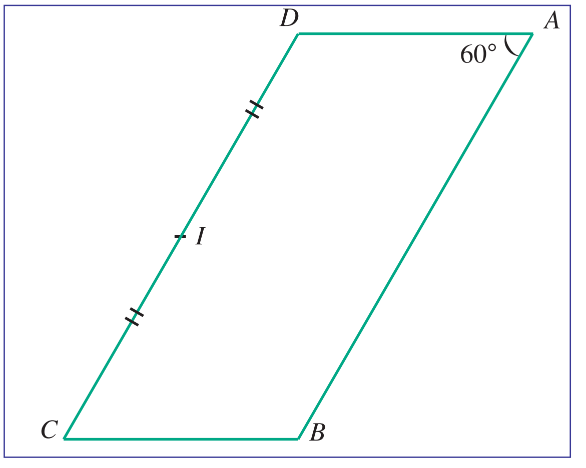
ABCD is a trapezoid. [*AB*], the small base, is equal to 4 cm, and [*CD*], the big base, is equal to 6 cm. *I, J, K* and *L* are respectively the midpoints of [*AD*], [*BC*], [*AC*] and [*BD*].
Calculate *KL* and the midsegment of the trapezoid.

EXERCISES AND PROBLEMS

Test your knowledge

1 *ABCD* is an isosceles trapezoid of bases [*AB*] and [*CD*] such that : $\widehat{BAD} = 60°$.
Calculate the other angles of this trapezoid.

2 *ABCD* is a parallelogram such that : $AB = 2AD$, $\widehat{BAD} = 60°$ and *I* is the midpoint of [*CD*].

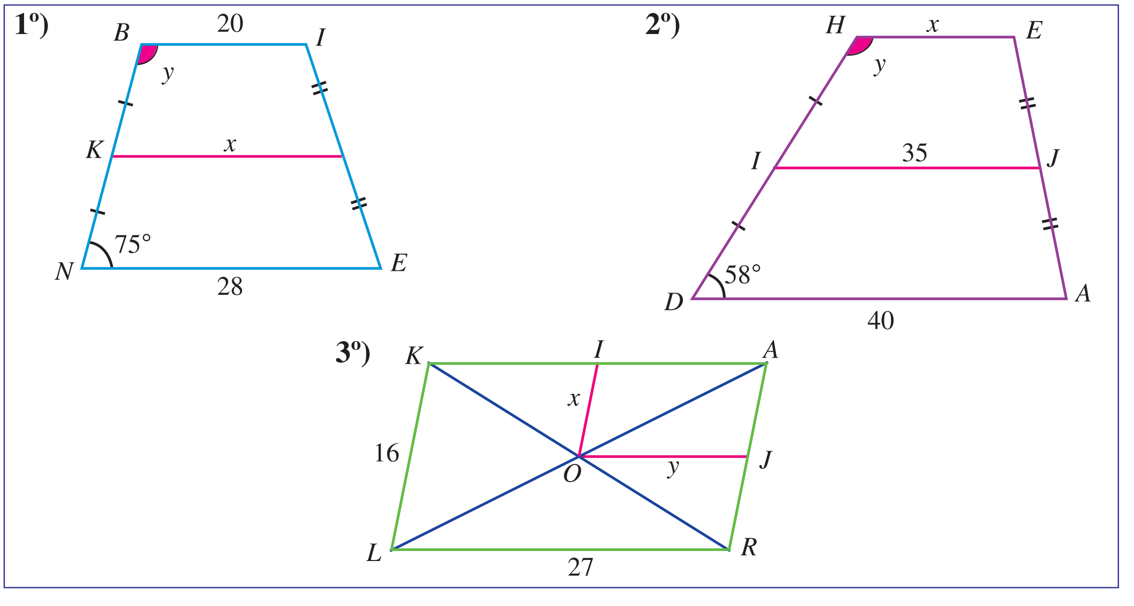


1°) What is the nature of triangle *BIC* ?

2°) Prove that *ABID* is an isosceles trapezoid.

3 *ABC* is an equilateral triangle having a perimeter of 12 cm. *M*, *N* and *P* are respectively the midpoints of [*AB*], [*AC*] and [*BC*].
Calculate the perimeter of triangle *MNP*.

4 In each of the following figures, find the value of *x* and the value of *y*.



5 *ABCD* is a parallelogram. *I* is the midpoint of [*CD*] and *A′* is the symmetric of *A* with respect to *I*.

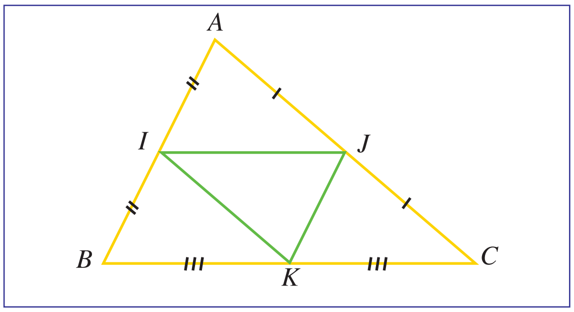
1º) Prove that *A′* , *C* and *B* are collinear.
What is the position of *C* on [*A′B*] ?

2º) If *O* is the center of the parallelogram *ABCD*, prove that $OI = \frac{1}{4} BA'$.

6 Observe the adjacent figure.

Name all the parallelograms having for vertices the points marked on the figure.

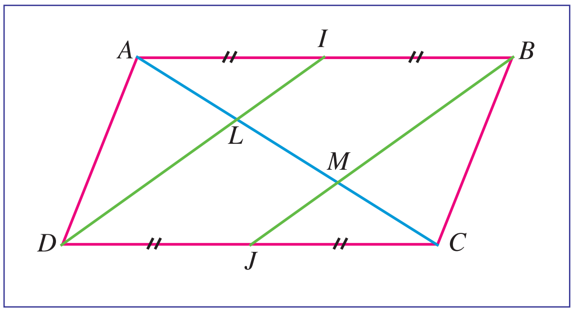
Justify your answer.



7 *ABCD* is a parallelogram.

1º) Prove that *IBJD* is a parallelogram.

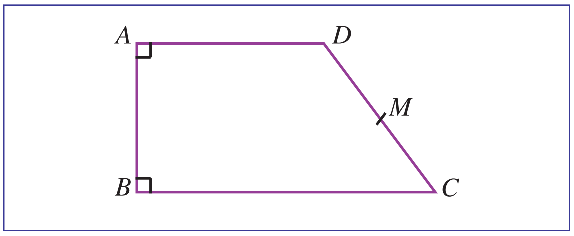
2º) Prove that $LM = \frac{1}{3} AC$.



8 *ABCD* is a right trapezoid of altitude *AB*.

Consider *M* the midpoint of [*CD*].

Prove that *MA* = *MB*.



9 *ABCD* is a quadrilateral. *E*, *F*, *G* and *H* are respectively the midpoints of [*AB*], [*BC*], [*CD*] and [*AD*].

Prove that quadrilateral *EFGH* is a parallelogram.

10 Consider a triangle *BEL*. *A* and *M* are respectively the midpoints of [*BE*] and [*EL*]. The segments [*BM*] and [*LA*] intersect at *O*.

R is the midpoint of [*BO*] and *I* the midpoint of [*LO*].

Prove that quadrilateral *RAMI* is a parallelogram.

11 *ABCD* is a rhombus. *M, N, P* and *Q* are respectively the midpoints of the sides [*AB*], [*BC*], [*CD*] and [*DA*].

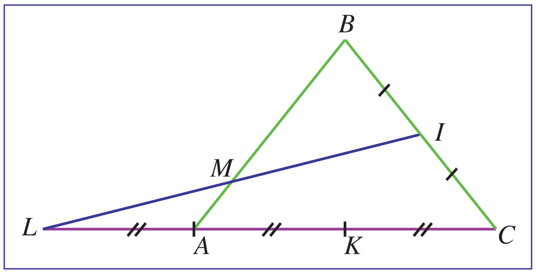
Prove that *MNPQ* is a rectangle.

12 [*AB*] and [*CD*] are respectively the small and the big base of a trapezoid *ABCD*. Extend the sides [*AD*] and [*BC*] until they meet at *E*; consider *M*, *N*, *P* and *Q* respectively the midpoints of [*AE*], [*BE*], [*AC*] and [*BD*]. Join *M* to *N* and *P* to *Q*.

Prove that *MNPQ* is a trapezoid.

13 *I* is the midpoint of [*BC*], *K* the midpoint of [*AC*] and *L* the symmetric of *K* with respect to *A*.

Prove that $AM = \frac{1}{4} AB$.



14 Answer by true or false.

1º) In a trapezoid, two sides are parallel.

2º) In an isosceles trapezoid, the nonparallel sides are congruent.

3º) In an isosceles trapezoid, the four angles are equal.

4º) In a trapezoid, the diagonals are congruent.

5º) In an isosceles trapezoid, the diagonals intersect at their midpoint.

6º) In a trapezoid, the bases have the same perpendicular bisector.

7º) In a trapezoid, the midsegment is the segment joining the midpoints of the bases.

8º) In a triangle, the segment having as extremities the midpoints of two sides is parallel to the third side and equals to its half.

9º) The midsegment in a trapezoid equals half the difference of the bases in this trapezoid.

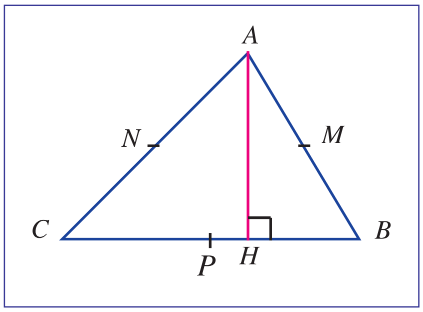
10º) In a triangle, the line drawn from the midpoint of a side and parallel to another, passes through the midpoint of the third side.

For seeking

15 *RAT* is a triangle. [*AM*] is the median and *O* is the midpoint of [*AM*]. (*RO*) cuts [*AT*] at *D*. Let *I* be the midpoint of [*TD*].

Prove that : $MI = \frac{1}{2} RD$, $TD = 2DA$ and $OD = \frac{1}{4} RD$.

16 *ABC* is a triangle. *M*, *N* and *P* are respectively the midpoints of [*AB*], [*AC*] and [*BC*]. [*AH*] is the height relative to [*BC*].

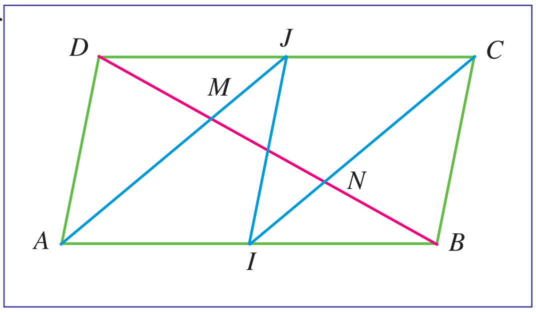


Prove that the points *M*, *N*, *P* and *H* are the vertices of an isosceles trapezoid.

17 *ABCD* is an isosceles trapezoid with [*AB*] and [*DC*] as bases. The bisectors of $\widehat{BAD}$ and $\widehat{ABC}$ intersect at *I*, and the bisectors of $\widehat{BCD}$ and $\widehat{ADC}$ intersect at *J*.

Prove that (*IJ*) is the axis of symmetry of *ABCD*.

18 *ABCD* is a parallelogram. *I* is the midpoint of [*AB*] and *J* is the midpoint of [*DC*]. Line (*BD*) cuts (*AJ*) at *M* and (*CI*) at *N*.



1°) Prove that *AICJ* is a parallelogram.

2°) Prove that *M* is the midpoint of [*DN*].

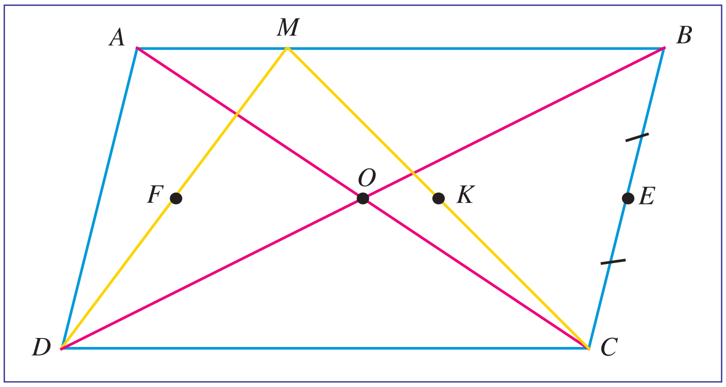
19 In a triangle *ABC*, [*AM*] is the median relative to [*BC*] and *O* is the midpoint of [*AM*]. (*BO*) cuts [*AC*] at *D*. Let *E* be the midpoint of [*DC*].

1°) Prove that *BMED* is a trapezoid.

2°) Prove that *D* is the midpoint of [*AE*]; deduce that $CD = 2AD$.

3°) Prove that $OD = \frac{1}{4} BD$.

20 *ABCD* is a parallelogram of center O such that $AB = 7$ cm and $AD = 4$ cm. M is a point of $[AB]$ such that $AM = x$. E and F are respectively the midpoints of $[BC]$ and $[DM]$. K is the midpoint of $[MC]$.

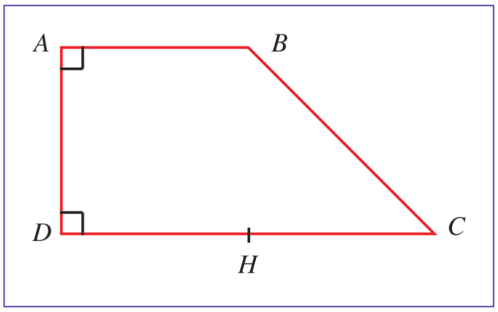


1º) Prove that E, K, O and F are collinear.

2º) **a)** Calculate EO.

b) Calculate OK and EF in terms of x.

21 *ABCD* is a right trapezoid. $[AD]$ is the height and $DC = 2AB = 2AD$.



Let H be the midpoint of $[DC]$.

1º) Prove that $[AH]$ and $[BD]$ intersect at their midpoint I.

2º) Calculate the angles of trapezoid *ABCD*.

3º) Prove that $[AC]$ and $[BH]$ intersect at their midpoint O.

4º) Calculate OI in terms of CD.

22 E, F, G and H are respectively the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ in a quadrilateral *ABCD*.

1º) Prove that *EFGH* is a parallelogram.

2º) In which case is *EFGH* :

a) a rectangle ?

b) a rhombus ?

c) a square ?

TEST

1 M, N and P are respectively the midpoints of $[AB]$, $[AC]$ and $[BC]$ in a triangle ABC.

Prove that the perimeter of triangle MNP equals half the perimeter of triangle ABC.

(6 points)

2 I and J are respectively the midpoints of $[AB]$ and $[AC]$ in a triangle ABC. Let E be the midpoint of $[BI]$ and F the midpoint of $[CJ]$.

Prove that $EF = \frac{3}{4} BC$. **(6 points)**

3 $ABCD$ is an isosceles trapezoid. $[AB]$ is the longer base. M and N are respectively the midpoints of $[BD]$ and $[AC]$. H and P are respectively the orthogonal projections of C and D on $[AB]$.

1º) Prove that $BH = AP$. **(4 points)**

2º) Prove that $MN = BH$. **(1 point)**

Deduce that $MNHB$ is a parallelogram. **(3 points)**

12 EQUATIONS OF THE FORM $(ax + b)(cx + d) = 0$

Objective

To solve the equations of the form of :

$(ax + b)(cx + d) = 0$

CHAPTER PLAN

COURSE

1. Reminder : solution of the equation $ax = b$
2. Equation of the form : $(ax + b)(cx + d) = 0$
3. Solution of the equations of the form of : $(ax + b)(cx + d) = 0$

EXERCISES AND PROBLEMS

TEST

COURSE

1 REMINDER : SOLUTION OF THE EQUATION $ax = b$

Activity

1°) Calculate y if $2y + 7 = 15$.

2°) Is $x = -4$ the solution of the equation : $3x + 5 = x - 3$?

3°) Calculate x if $2x + 4 = 2x + 1.5$.

4°) Does the equation $3x + 1 = 3\left(x + \frac{1}{3}\right)$ have a solution ?

Ruler

$ax = b$, with $a \neq 0$, admits $x = \frac{b}{a}$ as a solution

$0x = b$, with $b \neq 0$, admits no solution

$0x = 0$ admits all numbers as solutions

EXAMPLES

- $4x + 5 = x - 4$ is equivalent to $3x = -9$, where $x = -3$.
- $2x + 1 = 2(x + 2)$ is equivalent to $0x = 3$. This equation admits no solution.
- $3(x + 2) = 3x + 6$ is equivalent to $0x = 0$. This equation admits all numbers as solutions (infinite number of solutions).
- $\frac{2x}{3} - \frac{1}{2} = x - \frac{1}{3}$ is equivalent to $-\frac{1}{3}x = \frac{1}{6}$, so $x = -\frac{1}{2}$.

Application 1

Solve each of the following equations.

1°) $5.2(x + 0.5) = 2(x - 1.3)$. **2°)** $2\left(\frac{2x}{3} + 1\right) = \frac{4x}{3} + 9.5$. **3°)** $3(-x + 3) = -3x + 9$.

2 EQUATION OF THE FORM : $(ax + b)(cx + d) = 0$

Activity

Given the expression $F = (2x - 1)(x + 5)$.

1°) Solve the equation : $2x - 1 = 0$. Then complete : for $x = \frac{1}{2}$, $F = ...$

2°) Solve the equation : $x + 5 = 0$. Then complete : for $x = -5$, $F = ...$

Rule

If the **product of factors is equal to zero**, then it is sufficient that **one of the factors is equal to zero**.

To solve the equation $(ax + b)(cx + d) = 0$, solve the equation : $ax + b = 0$ and $cx + d = 0$.

As a result :

> $(ax + b)(cx + d) = 0$, with $a \neq 0$ and $c \neq 0$
>
> admits the solution : $x = -\dfrac{b}{a}$ and $x = -\dfrac{d}{c}$

EXAMPLE

Solve the equation : $(2x + 7)(-x + 4) = 0$.

The solution of this equation is $2x + 7 = 0$ or $-x + 4 = 0$, therefore $x = -\dfrac{7}{2}$ and $x = 4$.

Application 2

Solve each of the following equations : **1°)** $(2x - 5)(-x + 3) = 0$ **2°)** $x(3x + 5) = 0$.

3 SOLUTION OF THE EQUATIONS OF THE FORM : $(ax + b)(cx + d) = 0$

1°) $4x^2 - 7x = 0$.

Take **x as a factor** : $x(4x - 7) = 0$.

Thus the solution is $\boldsymbol{x = 0}$ **or** $\boldsymbol{x = \dfrac{7}{4}}$ **.**

2°) $(x - 5)(2x + 1) - (x - 5)(x + 4) = 0$.

Take **$(x - 5)$ as a factor** : $(x - 5)(2x + 1 - x - 4) = 0$,

$(x - 5)(x - 3) = 0$.

Thus the solution is $\boldsymbol{x = 5}$ **or** $\boldsymbol{x = 3}$**.**

3°) $x^2 = 9$.

This equation is written as : $x^2 - 9 = 0$,

$(x - 3)(x + 3) = 0$.

Thus the solution is $\boldsymbol{x = 3}$ **or** $\boldsymbol{x = -3}$**.**

4º) $(x+1)^2 = \frac{4}{9}$.

This equation is written as : $(x+1)^2 - \frac{4}{9} = 0$,

$$\left(x + 1 - \frac{2}{3}\right)\left(x + 1 + \frac{2}{3}\right) = 0 ,$$

$$\left(x + \frac{1}{3}\right)\left(x + \frac{5}{3}\right) = 0.$$

Thus, the solution is $\boldsymbol{x = -\frac{1}{3}}$ **or** $\boldsymbol{x = -\frac{5}{3}}$ **.**

5º) $x^2 + 7 = 0$.

This equation is written : $x^2 = -7$.

There is no solution because the square of any number cannot be strictly negative.

Application 3

Solve each of the following equations.

1º) $3x^2 - 8x = 0$.

2º) $\left(\frac{3}{5}x + 2\right)(3x-1) - \left(\frac{3}{5}x + 2\right)(x-4) = 0$.

3º) $x^2 = -5$.

4º) $(x+2)^2 = \frac{25}{16}$.

EXERCISES AND PROBLEMS

Test your knowledge

1 Verify whether the given number is a solution of the given equation.

1º) $x^2 + 2x + 1 = 0$; -1.

2º) $(a-5)(2a+1) = 0$; 4.

3º) $\frac{y^2}{2} = 2$; -2.

4º) $(z+1)^2 + 4 = 0$; 3.

5º) $x^2 + 9 = 0$; -3.

2 Solve each of the following equations.

1º) $2x^2 + 3x = 0$.

2º) $\left(\frac{2t}{3} + 1\right)\left(\frac{4}{5} - t\right) = 0$.

3º) $r^2 - 5 = 0$.

4º) $u^2 - 4u + 4 = 0$.

5º) $(y+5)^2 = 4$.

6º) $9t^2 + 6t + 1 = 0$.

3 Solve each of the following equations.

1º) $2y^2 - 8 = 0$. **2º)** $5x^2 + 1 = 4x^2 + 10$. **3º)** $2(x^2 - 7) = 3(x^2 - 5)$.

4º) $2(x^2 + 1) = x^2 - 3$. **5º)** $3x^2 = 24$. **6º)** $4x^2 - 6 = 3x^2 + 10$.

7º) $(x - 7)^2 + \frac{11}{3} = 0$. **8º)** $\frac{25}{49} = (x + 3)^2$.

4 Factorize then solve each of the following equations.

1º) $(x-5)(3x-4)-(2x+8)(x-5)=0$. **2º)** $(x - 2)(5x - 6) - 3x(4x - 8) = 0$.

3º) $x^2 - 4 = (x + 2)(3x - 10)$. **4º)** $\left(\frac{3x}{7} + 5\right)^2 - \left(\frac{x}{14} - 3\right)^2 = 0$.

5º) $(7x + 12)(x + 3) = x^2 + 6x + 9$. **6º)** $x^2 - 4x + 4 = (4x + 9)(x - 2)$.

5 Given $A(x) = x^2 - 6x + 5$.

1º) **a)** Factorize the expression $A(x) - 5$.

b) Solve the equation $A(x) = 5$.

2º) **a)** Develop and reduce the expression $(x - 3)^2 - 4$.

b) Factorize $A(x)$.

c) Then solve $x^2 - 6x + 5 = 0$.

3º) Solve the equation $A(x) = (x - 1)$.

For seeking

6 Find all the numbers x :

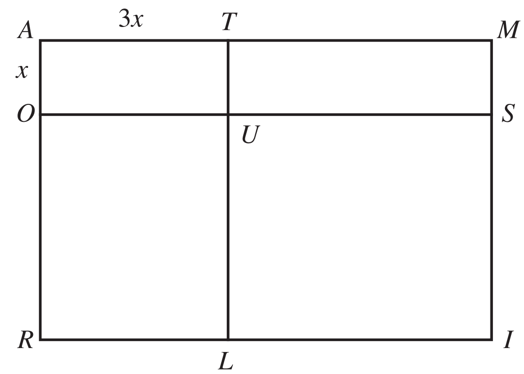
1º) equal to their reciprocal.
2º) equal to their square.

7 If x is subtracted from the side of a square of 6 cm length ($x > 6$), the area of the new square is $25\,\text{cm}^2$. Calculate x.

8 A rectangular field has as dimensions : 49 m and 36 m. Calculate the length of the side of a square having the same area as the field.

9 **1°)** Develop $S(x) = (x - 1)(x + 4)$.

2°) The adjacent figure represents a rectangle $AMIR$ of length $AM = 6$ cm and of width $MI = 4$ cm.
The segments $[TL]$ and $[OS]$ are such that : (TL) is parallel to (MI) and (OS) is parallel to (AM).
$AO = x$ and $AT = 3AO$.



a) Can the value of x be negative ?
b) Calculate, in terms of x, the area of the rectangles $OATU$ and $LUSI$.
c) Calculate x if the area of the rectangle $LUSI$ is three times that of $OATU$.

TEST

1 Solve each of the following equations. **(6 points)**

1°) $(x - 5)\left(3x + \frac{4}{3}\right) - \left(x + \frac{3}{2}\right)(5 - x) = 0.$

2°) $x^2 - 6x + 9 + (x - 3)(x + 4) = 0.$

3°) $(2y - 3)(y - 2) - 4y^2 + 12y - 9 = 0.$

4°) $\left(\frac{3t}{5} - \frac{2}{3}\right)^2 = \left(\frac{t}{2} - \frac{1}{2}\right)^2.$

5°) $6x^2 + 4 = 5x^2 + 9.$

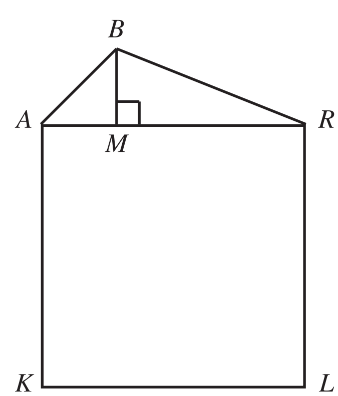
6°) $2x^2 + 1 = 3x^2 + 5.$

2 What is the length c of a side of a square having the same area as a disc of radius 2m ? (Use $\pi = 3.14$) **(2 points)**

3 Find a number x such that four times this number is equal to its square. Give all the solutions. **(2 points)**

4 Given $A(x) = 4x^2 + 8x - 5$. **(5 points)**
1°) **a)** Develop $(2x + 2)^2$.
b) Calculate m for having $A(x) = (2x + 2)^2 - m$.
2°) Solve the equation $A(x) = 0$.
3°) Solve the equation $A(x) = (2x + 5)$.

5 The unit of length is the centimeter. **(5 points)**
$KARL$ is a square. $MR = 7$ cm , $AM = BM = x$.



1°) Can the value of x be negative ?

2°) Find, in terms of x, the area A_1 of triangle BAR and the area A_2 of the square $KARL$.

3°) Calculate x if $A_2 = 6A_1$.

13 FIRST DEGREE INEQUALITY IN ONE UNKNOWN

Objectives

1. To use the properties of the operations on the order of decimal numbers.

2. To solve first degree inequalities in one unknown.

CHAPTER PLAN

COURSE

1. Comparing two numbers
2. Properties of inequalities
3. First degree Inequality in one unknown
4. Representation of the solution of an inequality on a number line
5. Problems written as first degree inequalities

EXERCISES AND PROBLEMS

TEST

COURSE

1 COMPARING TWO NUMBERS

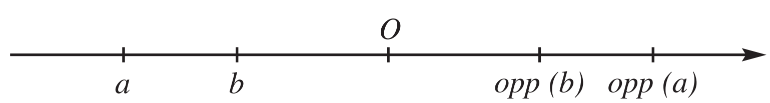
⊙ **To compare** the two numbers a and b is to say that a is **greater than** b, a is **equal** to b, or a is **less than** b.

Comparison	Read
$a < b$	a is inferior to b or a is less than b
$a \leqslant b$	a is less than or equal to b
$a > b$	a is superior to b or a is greater than b
$a \geqslant b$	a is greater than or equal to b
$a < 0$	a is strictly negative or a is less than 0
$a \leqslant 0$	«a is negative» or «a is less than or equal to 0»
$a > 0$	a is strictly positive or a is greater than 0
$a \geqslant 0$	«a is positive» or «a is greater than or equal to 0»

EXAMPLES

$-3.4 < 2.15$; $-5.12 < 0$; $4.7 > 0$.

⊙ **Two numbers are arranged in the inverse order of their opposites.**



If $a < b$ then opp $(a) >$ opp (b)

If $a < b$ then $-a > -b$

EXAMPLES

$2 < 5$ then $-2 > -5$; $-3 < 4$ then $3 > -4$; $-9 < -5$ then $9 > 5$.

Application 1

Compare : 4.13 and 4.103 ; – 32 and 5.3 ; – 15.4 and 0 ; 1.2 and 0 ; – 4.53 and – 2.

2 PROPERTIES OF INEQUALITIES

Activity

Observe and complete the following chart.

		Comparison
$\boldsymbol{a = -2}$	$\boldsymbol{b = 4}$	$\boldsymbol{a < b}$
$a + 3 = -2 + 3 = 1$	$b + 3 = 4 + 3 = 7$	$a \mathbf{+ 3} \mathbf{<} b \mathbf{+ 3}$
$a - 5 = \ldots$	$b - 5 = \ldots$	$a \mathbf{- 5} \ldots b \mathbf{- 5}$
$3a = \ldots$	$3b = \ldots$	...
$-2a = \ldots$	$-2b = \ldots$	...
$\frac{a}{2} = \ldots$	$\frac{b}{2} = \ldots$	...
$\frac{a}{-4} = \ldots$	$\frac{b}{-4} = \ldots$	...

1°) The order is preserved when adding or subtracting the same number on both sides of the inequality.

If $a \leqslant b$ then for every number c,

$$a + c \leqslant b + c$$
$$a - c \leqslant b - c$$

EXAMPLES

$3 < 6$ then $3 \mathbf{+ 5} < 6 \mathbf{+ 5}$ $(8 < 11)$.

$2 < 5$ then $2 \mathbf{- 3} < 5 \mathbf{- 3}$ $(-1 < 2)$.

2°) The order is preserved when multiplying or dividing by the same strictly positive number both sides of the inequality.

If $a \leqslant b$ then for every number $c > 0$,

$$a \times c \leqslant b \times c$$
$$a \div c \leqslant b \div c$$

EXAMPLES

$2 < 4$ then $2 \times \mathbf{3} < 4 \times \mathbf{3}$ $(6 < 12)$.

$10 < 14$ then $10 \div \mathbf{2} < 14 \div \mathbf{2}$ $(5 < 7)$.

3º) The order is reversed when multiplying or dividing by the same strictly negative number both sides of the inequality.

If $a \leqslant b$ then for every number $c < 0$,
$$a \times \boldsymbol{c} \geqslant b \times \boldsymbol{c}$$
$$a \div c \geqslant b \div c$$

EXAMPLES

$4 < 6$ then $4 \times \mathbf{(-2)} > 6 \times \mathbf{(-2)}$ $\quad (-8 > -12)$.

$6 < 8$ then $6 \div \mathbf{(-2)} > 8 \div \mathbf{(-2)}$ $\quad (-3 > -4)$.

Application 2

Write the obtained inequality :

1º) Add 3 to the two sides of $-4 < 11$, and $-3 > -6$.

2º) Subtract 7 to the two sides of $19 > 16$, and $-7 < 5$.

3º) Multiply by 2 the two sides of $-3 < 4$, and $-8 < -3$.

4º) Multiply by -3 the two sides of $-4 < -1$, and $13 > 2$.

5º) Divide by 3 the two sides of $6 < 15$, and $12 > 9$.

6º) Divide by -2 the two sides $-14 < 10$, and $18 > 16$.

3 FIRST DEGREE INEQUALITY IN ONE UNKNOWN

1º) Definition

A **first degree inequality in x** is **of the form** :

$\boldsymbol{ax + b > 0}$; $\boldsymbol{ax + b \geqslant 0}$; $\boldsymbol{ax + b < 0}$; $\boldsymbol{ax + b \leqslant 0}$,

where $\boldsymbol{x}$ is the **unknown**; a and b are two given numbers with $\boldsymbol{a \neq 0}$**.**

EXAMPLES

$2x + 3 > 0$; $-5x + 2 \geqslant 0$; $\dfrac{3x}{2} + 5 < 0$ and $-7.3x + 10 \leqslant 0$ are inequalities.

Remark

Let the inequality be : $ax + b \geqslant 0$.

Subtract b from both sides : $ax + b \mathbf{- b} \geqslant 0 \mathbf{- b}$,

$ax \geqslant - b$.

$ax + b \geqslant 0$ and $ax \geqslant - b$ are two equivalent inequalities. The term $\boldsymbol{b}$ is **transfered** from the first side to the second provided that its **sign is changed.**

2º) Resolution

⊙ The solution of an inequality in x is all values of x that verify the inequality.

⊙ Solve the given inequality to find all the solutions.

⊙ Two equivalent inequalities have the same solution.

EXAMPLES

Solve the following inequalities.

1º) $2x - 3 \geqslant 0$.

– **Isolate the term $\boldsymbol{x}$** by transfering $- 3$: $2x \geqslant 3$.

– To find x, we **divide both sides by 2**.

2 is positive, so we obtain $x \geqslant \frac{3}{2}$.

$\frac{3}{2}$ and every number greater than $\frac{3}{2}$ are solutions of the inequality $2x - 3 \geqslant 0$.

2º) $- 3x + 6 < 9$.

This inequality is written as : $- 3x < 9 \mathbf{- 6}$; then : $- 3x < 3$.

We divide both sides by $- 3$ (it is negative), so we obtain :

$x > \frac{3}{-3}$; therefore $x > - 1$.

Every number greater than $- 1$ is a solution of the inequality $- 3x + 6 < 9$.

3°) In particular

⊙ To solve the inequality : $3x + 1 > 3\,(x + 2)$.

This inequality is written : $3x + 1 > 3x + 6$,

$3x - 3x > 6 - 1$,

$0x > 5 \quad (0 > 5)$.

The result is impossible; there is no value for x that verifies this inequality.

⊙ To solve the inequality : $2x - 5 > 2x - 8$.

It is written as : $2x - 2x > -8 + 5$,

$0x > -3 \quad (0 > -3)$.

The result is always verified; all the values of x are solutions of this inequality.

⊙ To solve the inequality : $5x + 12 \leqslant 5\,(x + 3) - 3$.

It is written as : $5x + 12 \leqslant 5x + 12$,

$5x - 5x \leqslant 12 - 12$,

$0x \leqslant 0$.

The result is always verified; all the values of x are solutions of this inequality.

⊙ To solve the inequality : $7x + 8 > 7\,(x + 2) - 6$.

It is written as : $7x + 8 > 7x + 14 - 6$,

$7x - 7x > 8 - 8$,

$0x > 0$.

The result is impossible; there is no value for x that verifies the inequality.

Application 3

Answer by true or false.

1°) $0x < -4$ admits no solution.

2°) $0x > -3$ admits no solution.

3°) $0x < 7$ admits all values of x as a solution.

4°) $0x < 0$ admits all values of x as a solution.

5°) $0x > 0$ admits no solution.

4º) Chart summarizing the solution of first degree inequalities.

Inequality	Equivalent inequality	Solution	
		$a > 0$	**$a < 0$**
$ax + b \geqslant 0$	$ax \geqslant -b$	$x \geqslant \frac{-b}{a}$	$x \leqslant \frac{-b}{a}$
$ax + b > 0$	$ax > -b$	$x > \frac{-b}{a}$	$x < \frac{-b}{a}$
$ax + b \leqslant 0$	$ax \leqslant -b$	$x \leqslant \frac{-b}{a}$	$x \geqslant \frac{-b}{a}$
$ax + b < 0$	$ax < -b$	$x < \frac{-b}{a}$	$x > \frac{-b}{a}$

Application 4

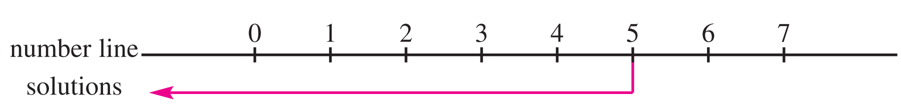
Solve each of the following inequalities.

1º) $x + 4 \leqslant 5$. **2º)** $-\frac{3}{2}x + 1 < x - \frac{3}{4}$. **3º)** $x - 5 > 2(2x - 1)$.

4º) $-5x + 3 \geqslant -7$. **5º)** $x\sqrt{3} - 1 > 2$. **6º)** $-3x - 4 < 3(+x - 2)$.

4 REPRESENTATION OF THE SOLUTION OF AN INEQUALITY ON A NUMBER LINE

⊙ The solutions of the inequality $x \leqslant 5$ are given by the following representation :

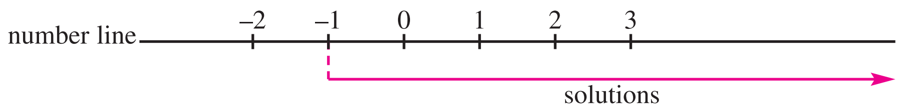


Since 5 is a solution, draw a complete line on 5.

⊙ Let the representation on the number line be the solution of the inequality $\frac{3}{2}x + \frac{4}{3} > \frac{x}{2} + \frac{1}{3}$.

This inequality is written $\frac{3}{2}x - \frac{x}{2} > \frac{1}{3} - \frac{4}{3}$

$x > -1$.



Since – 1 is not a solution, draw a dotted line on –1.

Application 5

Represent, on a number line, the solution of each of the following inequalities.

1º) $4x - 3 \geqslant 2x + 1$. **2º)** $\frac{4}{9}x - 1 < \frac{x}{3} - \frac{10}{9}$.

5 PROBLEMS WRITTEN AS FIRST DEGREE INEQUALITIES

The solution of a problem is often easier by performing an inequality.

For that, follow the following four steps :

1º) Choose the unknown (after reading and analyzing the text).

2º) Express the inequality.

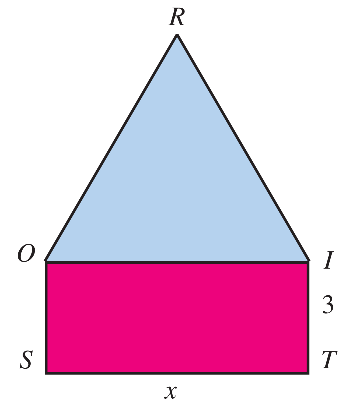
3º) Solve the inequality.

4º) Answer the problem.

EXAMPLE

The unit of length is the centimeter.

The adjacent figure represents a rectangle *SOIT* of width 3 and an equilateral triangle *ROI*.



Let the perimeter of the rectangle *SOIT* be greater than that of triangle *ROI*.

The problem is to determine the length of this rectangle.

Let x be its length, then :

the perimeter of *SOIT* is $2\ (x + 3)$, and the perimeter of *ROI* is $3x$.

The inequality is : $2\ (x + 3) < 3x$

$$2x + 6 < 3x$$

$$2x - 3x < -6$$

$$-x < -6$$

$$x > 6.$$

All the rectangles whose lengths are greater than 6 cm are answers to the problem.

These values ($x > 6$) are acceptable, since it is greater than the width 3 cm.

EXERCISES AND PROBLEMS

Test your knowledge

1 Complete the following table by adding or by subtracting the given number to both sides of the inequality.

Inequality	Inequality obtained by adding 3	Inequality obtained by subtracting 5
$a > 3$		
$b \leqslant -4$		
$c \geqslant 8$		

2 Complete the following table by multiplying or dividing the given number to both sides of the inequality.

Inequality	Inequality obtained by multiplying by 2	Inequality obtained by multiplying by – 3	Inequality obtained by dividing by 3	Inequality obtained by dividing by – 4
$x < -3$				
$y > 4$				
$z \leqslant -8$				

3 Represent, on a number line, the solution of each of the following inequalities.

1º) $x \geqslant 2$. **2º)** $x < -1$. **3º)** $x \leqslant 3$. **4º)** $x > -4$.

4 Among the numbers : – 6 ; – 8 ; 0 ; 1 and 4 , give those that are solutions of the inequality in t : $\frac{t}{2} + 5 > 3$.

5 Relate the equivalent inequalities.

$2x - 3 > 0$ •	• $x < \frac{-2}{\sqrt{3}}$
$\frac{-x}{3} + 5 < 3$ •	• $3x \leqslant -8$
$\frac{3x}{2} + 4 \leqslant 0$ •	• $x > \frac{3}{2}$
$-\sqrt{3}\,x > 2$ •	• $\frac{x}{3} > 2$

6 List the natural numbers that verify each of the inequalities :

1°) $x - 1 \leqslant 4.$ **2°)** $\dfrac{-x}{2} + 1 > -3.$

7 Solve each of the following inequality and represent the solution on a number line.

1°) $4x > -12.$ **2°)** $2x + 1 < -3.$

3°) $x + 5 \geqslant 2x - 1.$ **4°)** $\dfrac{x}{2} + 5 > x - \dfrac{1-x}{3}.$

5°) $13 - 4(2 + x) - 8x \leqslant -9.$ **6°)** $2 - \dfrac{x-3}{4} \geqslant \dfrac{x}{2} - \dfrac{1-x}{3}.$

7°) $-\dfrac{1}{4}x - 1 > \dfrac{1}{3}.$ **8°)** $-2x - 5\sqrt{2} \leqslant \sqrt{2}.$

9°) $\left(\dfrac{x}{2} - 1\right)^2 \leqslant \dfrac{x^2}{4} + x - 3.$ **10°)** $\dfrac{(1-x)^2}{2} - \dfrac{(x-1)}{3} < 2 + \dfrac{x^2}{2}.$

8 Answer by true or false.

1°) $\dfrac{13.1}{5} < \dfrac{13.01}{5}.$

2°) $\dfrac{10.3}{7} < \dfrac{10.3}{4}.$

3°) There exist five integers x such that : $-2 \leqslant x < 2.$

4°) A strictly positive number is always greater than its opposite.

5°) $-15.2 > -13.3.$

6°) $-\dfrac{1}{5} < -\dfrac{1}{7}.$

7°) If $a \leqslant 10$ then $a - 3 \leqslant 7.$

8°) If $b > -4$ then $-3b > 12.$

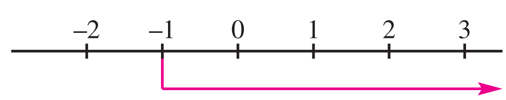
9°) $x = 0$ is a solution of the inequality $2x + 1 > 0.$

10°) $x = -1$ is a solution of the inequality $-x + 4 < 1.$

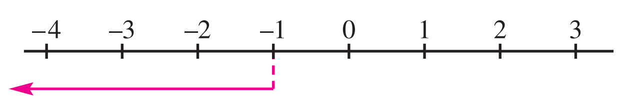
11°) The two inequalities $3x - 5 > 2$ and $3x > 7$ are equivalent.

12°) The two inequalities $-3x - 4 > 5$ and $-3x < 9$ have the same solution.

13°) The solution of the inequality $3x + 6 \geqslant 0$ is represented graphically by :



14°) The solution of the inequality $-\dfrac{x}{3} < 1$ is represented graphically by :



For seeking

9 Find the error in the following equivalent inequalities.

$\pi > 3$	$(3 - \pi)\,\pi > (3 - \pi)\,(3 + \pi)$
$3\,\pi > 9$	$\pi > 3 + \pi$
$3\,\pi - \pi^2 > 9 - \pi^2$	$0 > 3.$

10 A decimal number d is such that $d \leqslant -1.7$. Compare :

1°) $d + 31$ and 30.4.

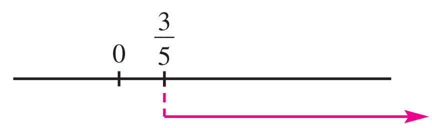
2°) $-3d$ and 5.

11 Write T (true) or F (false), for each case.

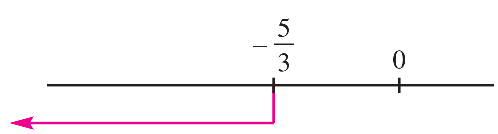
Inequality \ Number	-5	-1	$-\frac{1}{3}$	0	$\frac{5}{2}$	3
$3x + 1 < 0$						
$2x - 5 \geqslant 0$						
$5x + 1 > x - 3$						
$4 - x \leqslant 3 + 2x$						

12 Relate each inequality to the graphical representation of its solution.

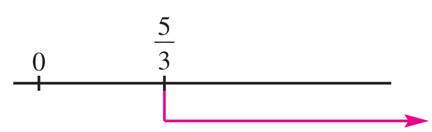
$3x - 1 \geqslant 4$



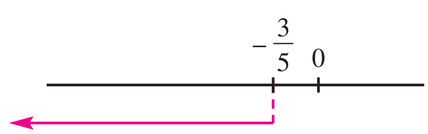
$3x + 1 < -2x - 2$



$12x - 5 > 2x + 1$



$\frac{x}{2} + \frac{5}{6} \leqslant 0$



13 Solve each of the following inequality and represent its solution on a number line.

1°) $3x - 7 - (x + 12) < 3$.

2°) $\frac{x}{5} - \frac{7}{3} + \frac{2x}{15} \geqslant 1$.

3°) $-5 + 3\left(\frac{x}{4} + 1\right) < \frac{3x}{2} + 1$.

4°) $\frac{8x + 3}{12} > 1 - \frac{4x - 1}{6}$.

5°) $\sqrt{2} - x\sqrt{2} < 6\sqrt{2}$.

6°) $\sqrt{2}\,(x + 3) \geqslant 3\sqrt{2}$.

7°) $5x + \sqrt{2} > 2\,(x + 1) + 3x$.

8°) $\left(\frac{x}{2} - 1\right)\left(\frac{x}{2} + 1\right) \geqslant x\left(\frac{x}{4} - 3\right)$.

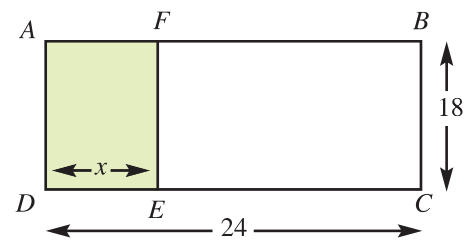
14 **1°)** Which integer when doubled is less than 5 ?
2°) Which number when tripled is greater than 4 ?
3°) Find a number whose third is less than or equal – 3 .

15 The triple of a natural number is greater than its half increased by 8.
1°) Let n be this number. Translate the given into an inequality.
2°) Solve this inequality.
3°) What are the possible values of n ?

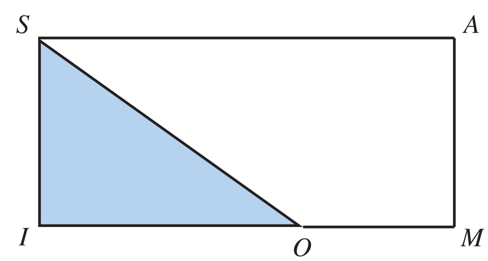
16 **1°)** If x is a natural number, what is the number that precedes it ? that follows it ?
2°) Translate into an inequality each of following statements :
⊙ the sum of three consecutive numbers is less than 9.
⊙ the sum of three consecutive numbers is more than 3.
3°) Solve each of these inequalities.
4°) Find all the natural numbers that verify the two preceding inequalities.

17 Marc scored on his first two mathematics assignments 11 and 10 over 20 . What should the third grade be so that the average of the three grades is less than or equal to 12 ?

18 The unit of length is the centimeter. Observe the adjacent figure. Find the value of x so that the area of $AFED$ is less than or equal to half the area of $ABCD$.



19 $SAMI$ is a rectangle such that $SA = 24$ cm and $SI = 10$ cm. O is a point on side $[IM]$ such that $IO = x$ cm.



2°) How does x vary ?
3°) Determine the position of O so that the area of triangle SOI is more than one third of the area of the rectangle $SAMI$.

20 A library offers the reader, for borrowing a book, the choice of two yearly price lists.
Price list 1 : yearly rental 450 000 L.L. then 5 000 L.L. for each rented book.
Price list 2 : 15 000 L.L. for rented each book.
For what number of books is the price list 1 more attractive ?

TEST

1 A decimal n is such that $n < -2.1$.

Compare : **(5 points)**

1º) $-5n$ and 10.

2º) $n - 4$ and -6.

2 Solve each of the following inequalities and represent the solutions on a number line.

1º) $\frac{3}{4}x + \frac{11}{2} > \frac{7x}{2} - \frac{5}{4}$. **(2 points)**

2º) $2x - 3(x+1)^2 \leqslant -2x^2 - 1 - (-x+3)^2$. **(2 points)**

3º) $\frac{-2x+2}{8} < 1 - \frac{5-x}{12}$. **(2 points)**

4º) $\sqrt{5}\,x + 10 \geqslant 2x + 5\sqrt{5}$. **(2 points)**

3 The unit of length is the centimeter.

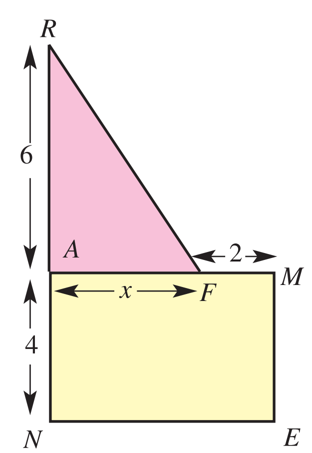
1º) Express, in terms of x :

a) the area of triangle RAF. **(2 points)**

b) the area of rectangle $AMEN$. **(2 points)**

2º) For what values of x is the area of triangle RAF less that half the area of rectangle $AMEN$?

(3 points)



14 PYTHAGOREAN THEOREM

Objective

To know and use the Pythagorean Theorem and its converse.

CHAPTER PLAN

COURSE

1. The Pythagorean Theorem
2. Converse of the Pythagorean Theorem
3. Height (altitude) of an equilateral triangle
4. Hypotenuse of an isosceles right triangle

EXERCISES AND PROBLEMS

TEST

1 THE PYTHAGOREAN THEOREM

Activity

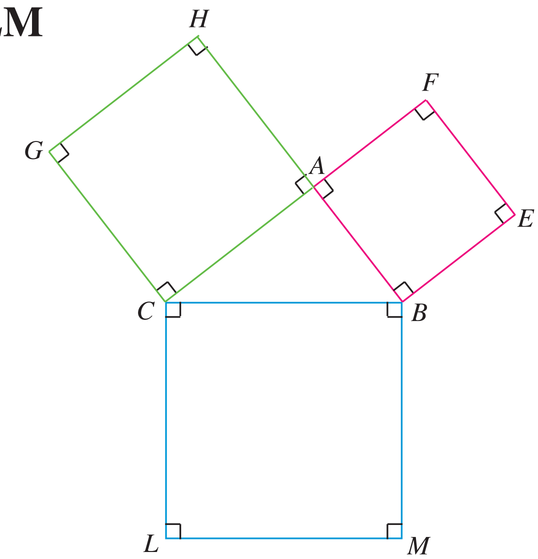
The length unit is the centimeter.

ABC is a right triangle at A such that :

$AB = 3$,

$AC = 4$ and

$BC = 5$.



Show that :

Area of square $ABEF$ + Area of square $ACGH$ = Area of square $BCLM$.

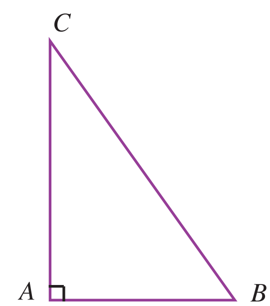
Deduce that $AB^2 + AC^2 = BC^2$.

The Pythagorean Theorem

In a right triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the two other sides (legs).

The adjacent triangle ABC is right at A.

Therefore : $BC^2 = AB^2 + AC^2$.



EXAMPLE

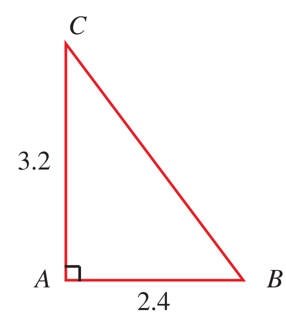
On this triangle, the Pythagorean theorem can be applied :

$BC^2 = AB^2 + AC^2$

$BC^2 = (2.4)^2 + (3.2)^2$

$BC^2 = 16.$

Thus $BC = 4$ cm.



Application 1

[BC] is the diameter of a circle C (O , 5 cm).

1º) On this circle, place a point A such that $BA = 8$ cm.

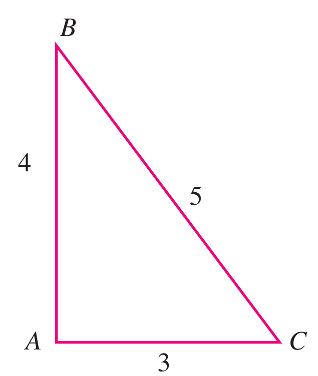
2º) Prove that triangle ABC is right.

3º) Calculate CA.

2 CONVERSE OF THE PYTHAGOREAN THEOREM

If the sides of a triangle ABC verify this relation :

$BC^2 = AB^2 + AC^2$, then triangle ABC is right at A.



In the adjacent triangle ABC, $AC = 3$, $AB = 4$ and $BC = 5$.

On one side : $AB^2 + AC^2 = 4^2 + 3^2 = 25$.

On the other side : $BC^2 = 5^2 = 25$.

Since : $4^2 + 3^2 = 5^2$, therefore $AB^2 + AC^2 = BC^2$.

Thus the triangle ABC is right at A.

Attention !

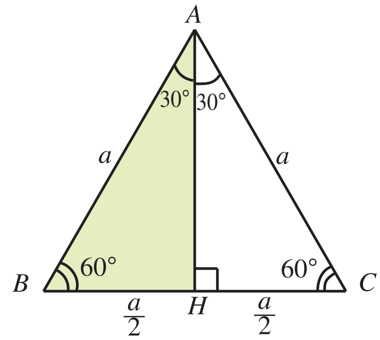
In a right triangle, the hypotenuse is the longest side.

Application 2

3.65 cm ; 0.27 cm and 3.64 cm are the lengths of the sides of a triangle.

Show that this triangle is right (use a calculator).

3 HEIGHT (ALTITUDE) OF AN EQUILATERAL TRIANGLE



Given an equilateral triangle ABC of side a. $[AH]$ is the height relative to $[BC]$.

Since triangle ABC is equilateral, the height $[AH]$ is also a median; therefore H is the midpoint of $[BC]$.

Thus : $BH = HC = \dfrac{a}{2}$.

Let's apply the pythagorean theorem to the right triangle ABH at H :

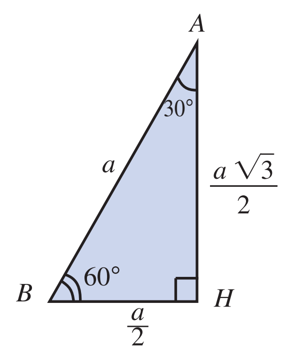
$$AH^2 + BH^2 = AB^2$$

$$AH^2 + \frac{a^2}{4} = a^2$$

$$AH^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}\ .$$

Thus : $\boxed{AH = \dfrac{a\sqrt{3}}{2}}$

Remark



A semi-equilateral triangle is a triangle having an angle of 30° , an angle of 60° and an angle of 90° .

In a semi-equilateral triangle :

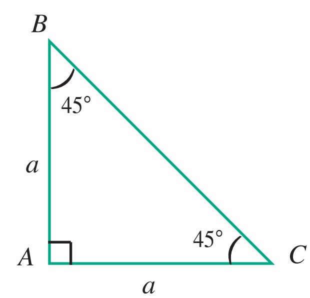
⊙ the side opposite to 30° equals half the hypotenuse $\left(BH = \dfrac{BA}{2}\right)$.

⊙ the side opposite to 60° equals the hypotenuse $\times \dfrac{\sqrt{3}}{2} \left(AH = \dfrac{BA\sqrt{3}}{2}\right)$.

Application 3

ABC is an equilateral triangle having a side of 6 cm. H is the foot of the perpendicular relative to $[BC]$. Calculate AH.

4 HYPOTENUSE OF AN ISOSCELES RIGHT TRIANGLE



Given an isosceles right triangle ABC of vertex A and such that $AB = AC = a$.

Apply the pythagorean theorem to triangle ABC which is right at A :

$$BC^2 = AB^2 + AC^2$$

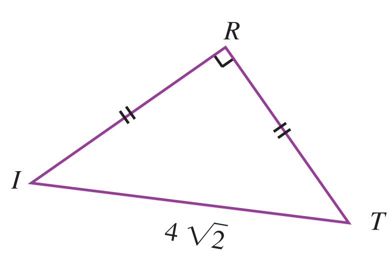
$$BC^2 = a^2 + a^2 = 2a^2$$

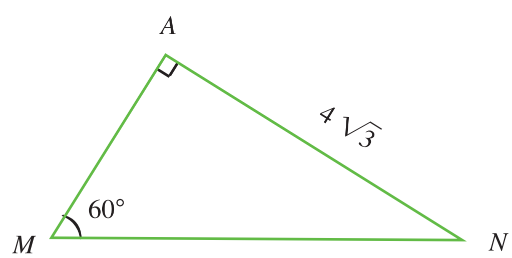
Therefore : $\boxed{BC = a\sqrt{2}}$

Application 4

1°) $ABCD$ is a square having each side equals to 5 cm. Calculate the length of its diagonal.

2°) Find the measures of the angles and the sides of triangles RIT and MAN.





EXERCISES AND PROBLEMS

Test your knowledge

1 BEL is a right triangle at E. M is the foot of the height relative to the hypotenuse.

Apply the pythagorean theorem to each of the following triangles : BEL, LEM and BEM.

2 $ABCD$ is a rectangle.

Prove that : $AB^2 + AD^2 = AC^2$.

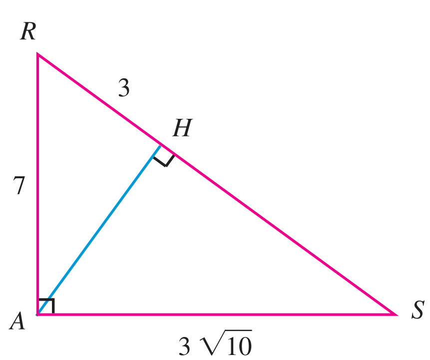
3 Calculate the diagonals of a rectangle $JOLI$ of dimensions 13 and 8; find the result to the nearest tenth (use your calculator).

4 Given RAS right at A and such that :

$RA = 7$, $AS = 3\sqrt{10}$ and $HR = 3$.

1º) Calculate the height AH.

2º) Calculate HS.



5 The diagonals of a rhombus measure 8 cm and 6 cm.

What is the length of the side of this rhombus ?

6 **1º)** Show that a triangle having sides equal to 10 , 8 and 6 , is a right triangle.

2º) If you double the sides, would you get a right triangle ?

7 Is a triangle, whose sides measure 2.1 ; 3 and 5.2 , a right triangle ? (Use a calculator).

If you double the sides, would you get a right triangle ?

8 **1º)** Is a triangle, whose sides measure 5.1 ; 6.8 and 8.5 , a right triangle? (Use a calculator).

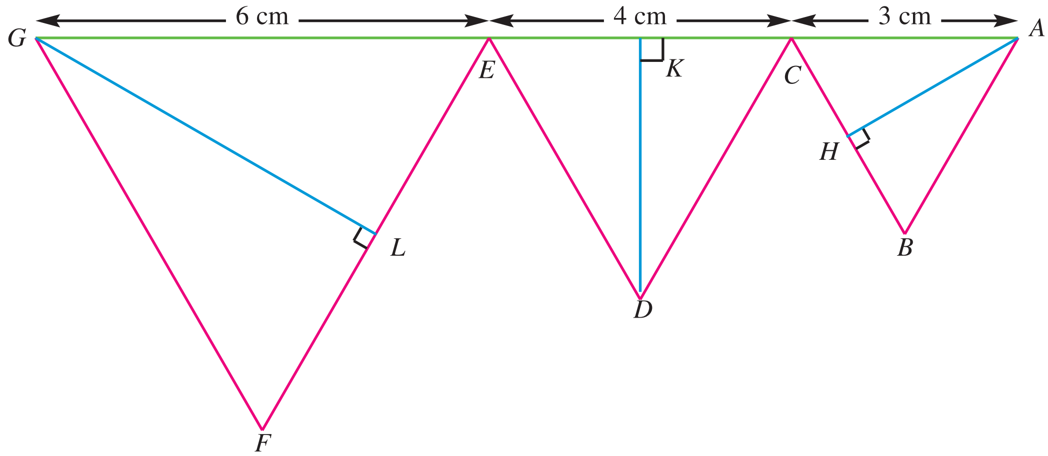
2º) If you double the sides, would you get a right triangle ?

9 Consider two triangles MAN and YES respectively right at A and E.

Given : $MA = \sqrt{7}$; $AN = 3$;
$YE = \sqrt{7} - 1$ and $ES = \sqrt{7} + 1$.

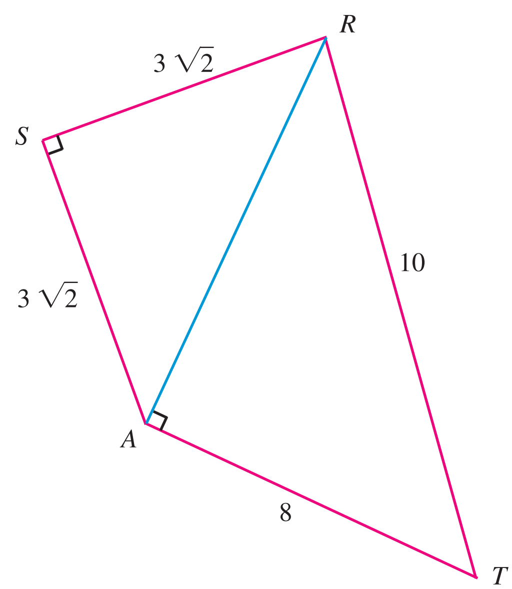
Show that $MN = YS$.

10 Calculate the heights AH, DK and GL of the following equilateral triangles :



11 In the adjacent figure, calculate AR.

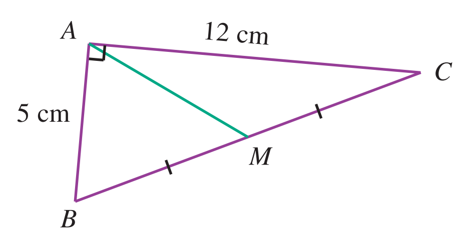
Then deduce that RAT is a right triangle.



12 ABC is a right triangle at A and such that $AB = 6$ cm.

Calculate AC if the area of this triangle is equal to 16.2 cm^2 .

13 What is the measure of the median $[AM]$ in the adjacent right triangle ?



14 Answer by true or false.

1º) A triangle with sides equal to 3 ; 4 and 5 is a right triangle.

2º) In a triangle EFG if :
$EF^2 = EG^2 + GF^2$; then this triangle is right at F.

3º) If $AB = 6$, and point M verifies : $MA^2 + MB^2 = 36$, then M is the vertex of a right triangle having $[AB]$ its hypotenuse.

4º) ABC is a right isosceles triangle of vertex A and such that $AB = AC = 5$. Therefore its hypotenuse $[BC]$ measures $5\sqrt{2}$.

5º) In an equilateral triangle having a side 5, the height measures $\frac{5\sqrt{3}}{2}$.

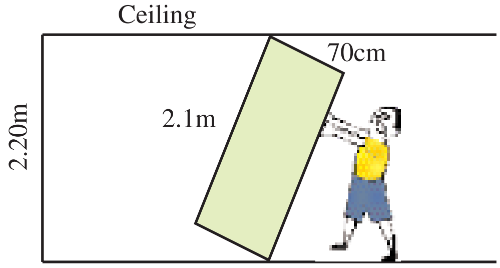
For seeking

15 $ABCD$ is a right trapezoid of height AB and such that : $AD = 7$ cm,
$BC = 4$ cm and $AC = 5.5$ cm.

Calculate the perimeter of this trapezoid.

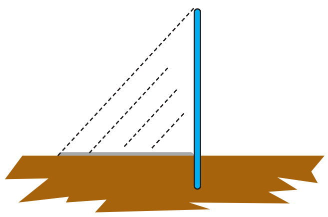
16 The dimensions of the closet are as follows : height 2.10 m; depth 70 cm.

Will this closet fit ?

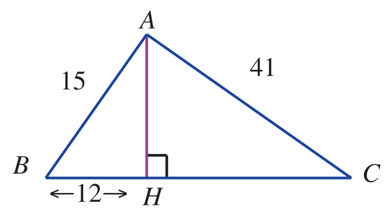


17 A 1.40 m long stick, sunk at 0.15 m deep in the ground, makes a shade of 0.90 m.

What is the distance between the superior extremity of the stick and the extremity of the shade ? (Round the answer to the nearest 0.01).

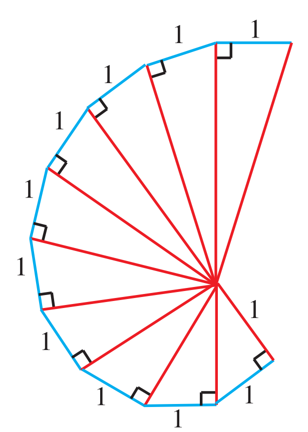


18 Using the information on the figure below, calculate the area of triangle ABC.

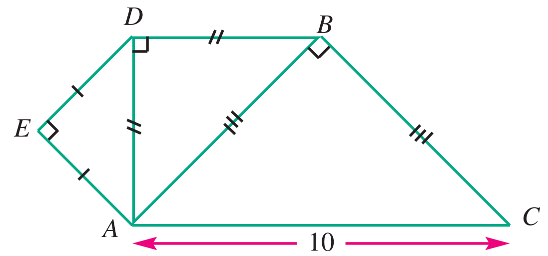


19 Each blue segment measures 1 cm.

Calculate the length of each red segment.



20 *ABC*, *ABD* and *AED* are right isosceles triangles having $AC = 10$.



1º) Calculate *AB*, *AD* and *AE*.

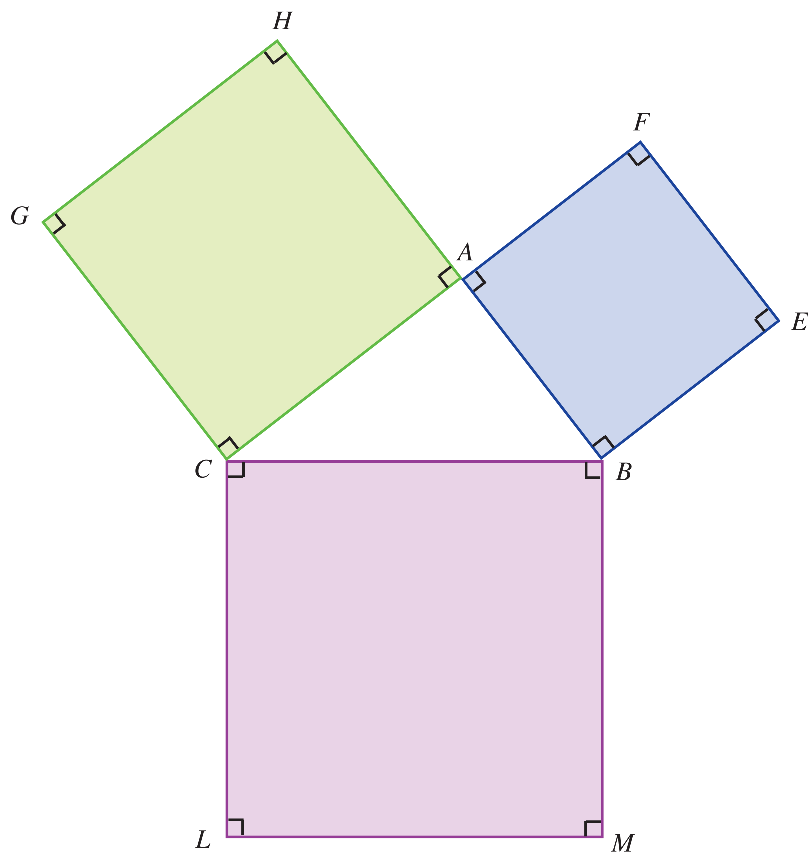
2º) Construct a right isosceles triangle *AFE* of vertex *F*. Calculate *AF*.

Are the points *F*, *A* and *C* collinear ?

Justify.

21 *ABEF*, *ACGH* and *BCLM* are three squares of respective areas 9 cm^2, 16 cm^2 and 25 cm^2.

Prove that *ABC* is a right triangle.



22 A television set «screen 90» has a screen whose diagonal equals 90 cm.

What is the measure of the side of this square television set «screen 90» ?

(Use a calculator).

TEST

1 *PAS* is a right triangle at *P* such that $PS = 12$ and $SA = 13$.

Calculate *PA*. **(3 points)**

2 *ARC* is a right triangle at *A* such that $CR = 10$ and $AC = 6$.

Calculate the lengths of the three medians in this triangle. **(6 points)**

3 *ABC* is a right isosceles triangle of vertex *B* and such that $BA = a$. *ACD* is a right triangle at *C* with $\widehat{CAD} = 30°$ and *D* is a point outside the triangle *ABC*.

Calculate *CD* and *AD*. **(6 points)**

4 *ABCD* is a square with sides equal 12 cm each. *P* is the midpoint of [*CD*] and *N* a point on [*BC*] such that $CN = 3$ cm.

Prove that *ANP* is a right triangle. **(5 points)**

15 FRACTIONAL EXPRESSIONS

Objective

To perform calculations on literal fractional expressions.

CHAPTER PLAN

COURSE

1. Literal fractional expression
2. Simplification of a literal fractional expression
3. Reducing to the same denominator
4. Addition and subtraction of fractional expressions
5. Multiplication of fractional expressions
6. Division of fractional expressions

EXERCISES AND PROBLEMS

TEST

COURSE

1 LITERAL FRACTIONAL EXPRESSION

Activity

1º) Calculate the numerical value of the expression $A = \dfrac{x+4}{x-2}$ for $x = -4$; $x = 5$.

2º) Can you find the numerical value of A for $x = 2$?

3º) What is the value of x that cancels the denominator of the expression $B = \dfrac{x-3}{x+4}$? Can you calculate the numerical value of B for $x = -4$?

Examples of literal fractional expressions

⊙ $\mathbf{\dfrac{4y+5}{y+7}}$, $\mathbf{\dfrac{3x+1}{x}}$ and $\mathbf{\dfrac{2}{3t-1}}$ are three **literal fractional expressions**, where x , y and t are variables.

⊙ A **literal fractional expression exists, is defined (has a meaning)** when **its denominator is different from zero.**

EXAMPLES

⊙ The literal fractional expression $\dfrac{x-4}{2x+1}$ is defined for :

$$2x + 1 \neq 0 \text{ , if } x \neq -\frac{1}{2} \text{ .}$$

⊙ The denominator of the literal fractional expression $\dfrac{m+2}{m^2-9}$ is not defined for : $m^2 - 9 = 0$; $(m-3)(m+3) = 0$, if $m = 3$ or $m = -3$.

For these values of the variable m, the given fractional expression is not defined.

Therefore, this expression exists for $m \neq 3$ and $m \neq -3$.

Application 1

Determine the values of the variable so that each of the following literal fractional expressions would be defined.

1º) $E = \dfrac{x+3}{-x+2}$. **2º)** $F = \dfrac{2y-1}{3y-18}$. **3º)** $G = \dfrac{m-7}{m^2-16}$.

2 SIMPLIFICATION OF A LITERAL FRACTIONAL EXPRESSION

To **simplify** a literal fractional expression :

Simplify :

$$E = \frac{m^2 - 2m + 1}{m^2 - 1}.$$

⊙ you **determine the values of the variables** so that **the expression would be defined.**

E is defined for $m^2 - 1 \neq 0$; $(m - 1)(m + 1) \neq 0$; if $m \neq 1$ and $m \neq -1$.

⊙ you **write**, if necessary, **the terms** (numerator and denominator) as **product of factors.**

Therefore E is written :

$$E = \frac{(m - 1)^2}{(m - 1)(m + 1)}.$$

⊙ you **cancel the common factors** of both terms, which means that you divide both terms by these factors.

If you simplify by $(m - 1)$, you obtain : $E = \dfrac{m - 1}{m + 1}$.

Application 2

Simplify each of the following literal fractional expressions.

1°) $\mathrm{E} = \dfrac{x^2 - 1}{(x + 1)^2}$.

2°) $\mathrm{F} = \dfrac{25\, a^6\, b^2\, c^4}{10\, a^4\, b\, c^5}$.

Remark

In some cases, **a change of signs would facilitate** the **simplification** of a fractional expression.

EXAMPLE

Simplify the expression : $\dfrac{m^2-4}{2-m}$.

This expression exists for $2-m \neq 0$, if $m \neq 2$.

Therefore, you can write : $\dfrac{m^2-4}{2-m} = -\dfrac{m^2-4}{m-2} = -\dfrac{(m-2)(m+2)}{m-2} = -(m+2)$.

3 REDUCING TO THE SAME DENOMINATOR

To **reduce** many fractional expressions to the same denominator :

Given the expressions :

$\dfrac{2x^2+xy}{x^2}$; $\dfrac{z^2-2xz}{yz}$; $\dfrac{x+2y}{z}$.

- you **simplify** these expressions, if necessary, after the conditions of existence have been determined.

For $x \neq 0$, $y \neq 0$ and $z \neq 0$, by simplifying, you obtain :

$\dfrac{2x+y}{x}$; $\dfrac{z-2x}{y}$; $\dfrac{x+2y}{z}$.

- you **choose** the **simplest common denominator**.

The common denominator is $x\,y\,z$.

- you **find the equivalent fractions.**

You get :

$\dfrac{yz(2x+y)}{xyz}$; $\dfrac{xz(z-2x)}{xyz}$; $\dfrac{xy(x+2y)}{xyz}$

Application 3

The fractional expressions : $\dfrac{2(a-1)}{a+2}$ and $\dfrac{3(a+2)}{4(a-4)}$ are respectively defined for $a+2 \neq 0$ and $a-4 \neq 0$; if $a \neq -2$ and $a \neq 4$, $4(a+2)(a-4)$ is their common denominator.

Therefore reduce these expressions to the same denominator.

4 ADDITION AND SUBTRACTION OF FRACTIONAL EXPRESSIONS

To **add** or **subtract fractional expressions** :

Simplify :

$$E = \frac{1}{m+1} - \frac{1}{m-1} + \frac{2m}{m^2-1} .$$

⊙ you **simplify these expressions after the conditions of existence have been determined;** then you **reduce them to the same denominator.**

For $m \neq 1$ and $m \neq -1$, we have :

$$E = \frac{m-1}{(m-1)(m+1)} - \frac{m+1}{(m-1)(m+1)} + \frac{2m}{(m-1)(m+1)}$$

⊙ you **add** or **subtract the numerators.**

$$E = \frac{m-1-(m+1)+2m}{(m-1)(m+1)}$$

$$= \frac{m-1-m-1+2m}{(m-1)(m+1)}$$

$$E = \frac{2m-2}{(m-1)(m+1)} .$$

⊙ you **simplify the answer**, if possible.

$$E = \frac{2(m-1)}{(m-1)(m+1)} = \frac{2}{m+1} .$$

Application 4

Calculate .

1°) $F = \frac{2a}{b} - \frac{4}{3a} + \frac{b}{3}$.

2°) $G = \frac{a-2}{a+2} + \frac{a+2}{a-2} - \frac{a^2+4}{a^2-4}$.

5 MULTIPLICATION OF FRACTIONAL EXPRESSIONS

To **multiply the fractional expressions** :

Calculate $P = \dfrac{a^2 - 9}{8a} \times \dfrac{6a}{a + 3}$.

- you **simplify these expressions**, if possible, after finding the condition of existence.

The existence conditions :
$a \neq 0$ and $a \neq -3$.

- you **multiply** the numerators, then the denominators.

$P = \dfrac{(a^2 - 9) \times 6a}{8a \times (a + 3)}$.

- If necessary,
  - you **decompose into factors both terms** of the obtained expression,
  - you **simplify**.

$P = \dfrac{(a - 3)\,(a + 3) \times 2 \times 3a}{2 \times 4a\,(a + 3)}$

$P = \dfrac{3\,(a - 3)}{4}$.

Application 5

Perform the following products.

1°) $\dfrac{2x}{x - 4} \times \dfrac{x^2 - 16}{8}$.

2°) $\dfrac{1 + 4x}{1 - 4x} \times \dfrac{(1 - 4x)^2}{(1 + 4x)^2}$.

3°) $\dfrac{a + 2}{a + 3} \times \dfrac{a^2 - 9}{a^2 - 4}$.

6 DIVISION OF FRACTIONAL EXPRESSIONS

To divide a fractional expression by a second one, you multiply the first by the reciprocal of the second one.

EXAMPLE

Calculate the quotient $Q = \dfrac{x^2 - 4}{3x} \div \dfrac{x - 2}{6x}$.

Q is written : $Q = \dfrac{x^2 - 4}{3x} \times \dfrac{6x}{x - 2}$. For $x \neq 0$ and $x \neq 2$, we have :

$Q = \dfrac{(x - 2)(x + 2)}{3x} \times \dfrac{6x}{x - 2}$

$Q = 2(x + 2)$.

Remark

When performing operations on Fractional expressions, if some parts of the expression are whole numbers or expressions without a denominator, consider their denominator to be 1 and apply the rules.

EXAMPLE

Calculate $E = 3m^2 + 1 - \dfrac{3m^2 n^2 + 1}{n^2}$.

$E = \dfrac{3m^2 + 1}{1} - \dfrac{3m^2 n^2 + 1}{n^2}$.

For $n \neq 0$, $\quad E = \dfrac{(3m^2 + 1)n^2 - (3m^2 n^2 + 1)}{n^2} = \dfrac{3m^2 n^2 + n^2 - 3m^2 n^2 - 1}{n^2}$

$E = \dfrac{n^2 - 1}{n^2}$.

EXERCISES AND PROBLEMS

Test your knowledge

1 Answer by true or false.

1°) The fractional expression $\frac{4+x}{5+x}$ is not defined for $x = -5$.

2°) The expression $\frac{x-2}{x+3}$ is defined for $x \neq 2$.

3°) The expression $\frac{2x+1}{x-4}$ is defined for any number x.

4°) The expression $\frac{5}{2+x^2}$ is defined for any number x.

5°) If x is different from 2, then $\frac{3(x-2)}{5(x-2)} = \frac{3}{5}$.

6°) The opposite of $2a - b$ is $-2a + b$.

7°) The reciprocal of $m^2 - 1$ is $\frac{1}{m^2-1}$.

2 Simplify each of the following fractional expressions.

1°) $\frac{x+x^2}{1+x}$.

2°) $\frac{2x^2+4x}{3x+6}$.

3°) $\frac{x^3+3x^2}{x^2-9}$.

4°) $\frac{2ax-10x}{a^2-25}$.

5°) $\frac{a^2+4a+4}{a^2-4}$.

6°) $\frac{4a^2-12a+9}{4a^2-9}$.

3 Perform the following operations.

1°) $\frac{m}{m+1} + \frac{n}{m+1}$.

2°) $\frac{x^2}{x+3} - \frac{9}{x+3}$.

3°) $\frac{m-1}{m} - \frac{m+1}{m-1}$.

4°) $\frac{1}{x-1} - \frac{2}{x-2}$.

4 Perform these products.

1°) $\frac{4x}{5y} \times \frac{25y}{8x}$.

2°) $\frac{5x^2y}{3y} \times \left(\frac{-3y^2}{5x^2}\right)$.

3°) $\left(\frac{-4a^2b}{9}\right) \times \left(\frac{-3c}{4b^2}\right)$.

4°) $\left(\frac{-3a^2}{2a}\right) \times \left(\frac{-2b^2}{3}\right) \times \left(\frac{-4}{5b}\right)$.

5 Reduce to the same denominator and calculate.

1°) $1 - \frac{x-3}{x+3}$.

2°) $a - 2 + \frac{4}{a+2}$.

3°) $\frac{1}{m-1} - \frac{1}{m+1} + 1$.

4°) $\frac{x}{x-5} + \frac{5}{5-x}$.

5°) $\frac{1}{a+1} + \frac{2}{a^2-1}$.

6°) $\frac{30x}{9x^2-1} + \frac{4}{3x-1} - \frac{5}{3x+1}$.

6 Divide.

1°) $\frac{x}{y} \div \frac{y}{x}$.

2°) $\frac{6x}{y} \div \frac{x}{3}$.

3°) $5ax \div \frac{10x}{b}$.

4°) $\frac{8-2x^2}{3x} \div (2+x)$.

5°) $(a^2-9) \div \left(\frac{1}{a} - \frac{1}{3}\right)$.

6°) $\left(\frac{1}{a^2} - 1\right) \div \left(\frac{1}{a} + 1\right)$.

For seeking

7 Given the two expressions.
$A(x) = 4\,(3x - 1)^2 - 25\,(-3x + 2)^2$ and
$B(x) = (7x-4)^2 - 2(4-7x)(4x-3) + 49x^2 - 16$.

1°) Factorize $A(x)$ and $(B(x)$.

2°) Given the fractional expression $F(x) = \dfrac{A(x)}{B(x)}$.

a) For which values of x does $F(x)$ exist ?

b) Simplify $F(x)$.

c) Solve $F(x) = -1$.

3°) Given $G(x) = \dfrac{B(x)}{A(x)}$.

a) For which values of x does $G(x)$ exist ?

b) Simplify $G(x)$.

8 Given the expression
$A(x) = \dfrac{1}{x+1} - \dfrac{3}{x-1}$.

1°) For which values of x does $A(x)$ exist ?

2°) Write $A(x)$ in the form of a fractional expression.

3°) Solve the equation $A(x) = 0$.

4°) Calculate $A\left(\dfrac{1}{2}\right)$.

9 Given the two expressions.
$P(x) = 9\,(2x - 1)^2 - 4$ and
$Q(x) = (6x - 5)\,(x - 2) - 12x + 10$.

1°) Factorize $P(x)$ and $Q(x)$.

2°) Solve the equation $P(x) = Q(x)$.

3°) Given $F(x) = \dfrac{P(x)}{Q(x)}$.

a) For which values of x does the expression $F(x)$ exist ?

b) Simplify $F(x)$.

c) Solve the equation $F(x) = -2$.

10 Given the expression
$A(x) = \dfrac{x-1}{2x-3} - \dfrac{x-2}{2x+1}$.

1°) For which values of x is $A(x)$ defined ?

2°) Write $A(x)$ in the form of a fractional expression.

3°) Calculate $A(0)$; $A(-1)$ and $A(0.5)$.

11 Given the two expressions.
$A(x) = (x - 2)\,(x + 5) - 2x + 4$ and
$B(x) = (2x - 3)^2 - (x - 1)^2$.

1°) Factorize $A(x)$ and $B(x)$.

2°) Solve $A(x) = 0$ and $B(x) = 0$.

3°) Given $F(x) = \dfrac{A(x)}{B(x)}$.

a) For which values of x is $F(x)$ defined ?

b) Simplify $F(x)$.

c) Calculate $F\left(\dfrac{1}{2}\right)$.

d) Solve the equation $F(x) = -2$.

12 Given the expression
$A(x) = -x - 4 - \dfrac{1}{x+2}$.

1°) For which value of x is $A(x)$ defined ?

2°) Calculate $A(0)$; $A(-1)$ and $A(-3)$.

3°) Solve the equation $A(x) = \dfrac{x+2}{4}$.

TEST

1 Simplify each of the following literal fractional expressions. **(4 points)**

1º) $\dfrac{x^2 - 4}{(x + 2)^2}$.

2º) $\dfrac{9a^2 - 12a + 4}{9a^2 - 4}$.

2 Perform the following operations. **(4 points)**

1º) $\dfrac{x^2}{x - 3} - \dfrac{9}{x - 3}$.

2º) $\dfrac{1}{m - 1} - \dfrac{1}{m + 1} - \dfrac{2m^2}{m^2 - 1}$.

3 Perform the following operations. **(4 points)**

1º) $\dfrac{a + 1}{a - 1} \times \dfrac{2a^2 - 4a + 2}{a^2 + 2a + 1}$.

2º) $\dfrac{a^2 - 4}{a^2 + 4a} \div \dfrac{a^2 - 2a}{a + 4}$.

4 Given the two expressions. **(8 points)**

$A(x) = (x^2 - 49) + (3x + 1)\,(x + 7)$ and

$B(x) = (x^2 + 14x + 49) + 3x\,(x + 7) + (x - 4)\,(2x + 14)$.

1º) Factorize $A(x)$ and $B(x)$.

2º) Solve $A(x) = 0$ then $A(x) = B(x)$.

3º) Given $F(x) = \dfrac{A(x)}{B(x)}$.

a) For which values of x is $F(x)$ defined ?

b) Simplify $F(x)$.

16 PROPORTIONALITY

Objectives

1. Solving problems involving directly proportional magnitudes.
2. Solving problems involving inversely proportional magnitudes.

CHAPTER PLAN

1 REMINDER

1°) Proportion

⊙ $\frac{a}{b}$ and $\frac{c}{d}$ are two equal ratios.

The equality $\frac{a}{b} = \frac{c}{d}$ called a proportion.

1st term — 3rd term

$$\boxed{\frac{a}{b} = \frac{c}{d}}$$

2nd term — 4th term

The first and the fourth terms are the **extremes**.

The second and the third terms are the **means**.

⊙ If $\frac{a}{b} = \frac{c}{d}$, therefore :

$a \times d = b \times c$ (the product of the means is equal to the product of the extremes),

$\frac{a}{c} = \frac{b}{d}$ (you permute the means),

$\frac{d}{b} = \frac{c}{a}$ (you permute the extremes),

$\frac{b}{a} = \frac{d}{c}$ (you inverse the two ratios).

Application 1

Write all the proportions that you can form using the following numbers : 2.5 ; 20 ; 25 and 200 .

2°) Directly proportional magnitudes

a) Two magnitudes x and y are **directly proportional** if you obtain the second numbers, «y» , upon multiplying the first numbers x by the same number K .

***K* is the proportionality coefficient.**

$$\boxed{y = Kx}$$

Remark

From $y = Kx$, we conclude $\boldsymbol{x = \frac{1}{K} y}$; therefore **x is directly proportional to y**.

EXAMPLES

x	3	4	6	10
y	7.5	10	15	25

$$\frac{7.5}{3} = \frac{10}{4} = \frac{15}{6} = \frac{25}{10} = \mathbf{\frac{5}{2}} .$$

$$\frac{y}{x} = \frac{5}{2} \quad ; \quad 2y = 5x \quad ; \quad y = \frac{5}{2}x \quad ;$$

$$\boxed{\boldsymbol{y = 2.5x}}$$

***y* is directly proportional to *x* ; 2.5 is the proportionality coefficient.**

b) «The numbers a, b and c are **directly proportional** to 4 ; 6 and 8» means that : $\boldsymbol{\frac{a}{4} = \frac{b}{6} = \frac{c}{8}}$.

Application 2

1º) In each of the following tables, are the magnitudes x and y directly proportional ? If yes, complete the following : $y = ... x$.

a)

x	1	0.1	0.2	0.3	0.5
y	65	6.5	13	19.5	32.5

b)

x	1.5	2	4	3	9
y	3	8	16	6	18

2º) Calculate the numbers a and b knowing that a, b and 6 are directly proportional to 2; 4 and 3.

2 INVERSELY PROPORTIONAL MAGNITUDES

a) Activity

1º) Complete the following tables.

x	2	6	9	13
y	6	18	27	39
$\frac{y}{x}$ (Irreducible fraction)				

z	2	3	4	6
t	18	12	9	6
$z \times t$				

2º) Complete.

The ratio of y to x is : $\frac{y}{x} = \ldots$, therefore $y = \ldots x$.

The product of z and t is : $z \times t = \ldots$, therefore $z = \frac{\ldots}{t}$.

3º) In the first table, are the numbers of the first row directly proportional to those of the second ? What is the proportionality coefficient (constant) ?

4º) In the second table, are the numbers of the first row directly proportional to those of the second ?

5º) Is the following table a table of proportionality ?

z	2	3	4	6
$\frac{1}{t}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{1}{6}$

b) Definition

Two magnitudes x and y are **inversely proportional** if there is a **constant number *K*** such that :

$$\boldsymbol{x \times y = K}\ .$$

If $x \neq 0$ and $y \neq 0$, from $x \times y = K$, we conclude that : $x = K \times \frac{1}{y}$ and $y = K \times \frac{1}{x}$; therefore, one of the magnitudes is directly proportional to the reciprocal of the other one.

EXAMPLES

1.

x	1	2	4
y	20	10	5

$1 \times 20 = 2 \times 10 = 4 \times 5 = 20.$

$\boldsymbol{x \times y = 20}$ **.**

The magnitudes x and y are inversely proportional.

2.

x	3	6	7
y	4	2	4

$3 \times 4 = 6 \times 2 \neq 7 \times 4$.

The magnitudes x and y are inversely proportional.

c) Remark

«The numbers a, b and c are **inversely proportional** to 4 ; 6 and 8» means that :

$\boldsymbol{4 \times a = 6 \times b = 8 \times c}$

or that : $\dfrac{a}{\frac{1}{4}} = \dfrac{b}{\frac{1}{6}} = \dfrac{c}{\frac{1}{8}}$.

Therefore, a, b and c are **directly proportional to the reciprocals** of 4; 6 and 8.

Application 3

1°) In each of the following cases, are the magnitudes x and y inversely proportional ? If yes, complete the following : $x \times y = \ldots$; $y = \dfrac{\ldots}{x}$.

a)

x	8	1	2	4
y	8	64	32	16

b)

x	15	30	6	10	8
y	2	1	5	3	5

2°) Calculate the numbers a and b knowing that a, b and 6 are inversely proportional to 2; 4 and 3.

3 SOLVED EXERCISES

1. Divide the number 60 into parts directly proportional to 2 ; 3 and 5.

If we designate these three parts by a, b and c, then :

$\frac{a}{2} = \frac{b}{3} = \frac{c}{5}$ and $a + b + c = 60.$

Suppose $\frac{a}{2} = \frac{b}{3} = \frac{c}{5} = t$.

$\frac{a}{2} = t$ gives $a = 2t$;

$\frac{b}{3} = t$ gives $b = 3t$;

$\frac{c}{5} = t$ gives $c = 5t$.

But : $a + b + c = 60.$

Therefore : $2t + 3t + 5t = 60$; $10t = 60$; $t = \frac{60}{10} = 6.$

Consequently : $a = 2t = 2 \times 6 = 12$;

$b = 3t = 3 \times 6 = 18$;

$c = 5t = 5 \times 6 = 30$.

2. Divide the number 61 into parts inversely proportional to 3; 4 and 7.

If we designate these three parts by a, b and c , then :

$3 \times a = 4 \times b = 7 \times c$ and $a + b + c = 61.$

Suppose $3 \times a = 4 \times b = 7 \times c = t.$

$3 \times a = t$ gives $a = \frac{t}{3}$;

$4 \times b = t$ gives $b = \frac{t}{4}$;

$7 \times c = t$ gives $c = \frac{t}{7}$.

But : $a + b + c = 61.$

Therefore : $\frac{t}{3} + \frac{t}{4} + \frac{t}{7} = 61$; $\frac{28t + 21t + 12t}{84} = 61$;

$\frac{61t}{84} = 61$; $61t = 61 \times 84$; $t = 84$.

Consequently : $a = \frac{t}{3} = \frac{84}{3} = 28$;

$b = \frac{t}{4} = \frac{84}{4} = 21$;

$c = \frac{t}{7} = \frac{84}{7} = 12$.

EXERCISES AND PROBLEMS

Test your knowledge

1 Calculate the unknown.

1º) $\dfrac{b+5}{b+2} = \dfrac{2}{5}$ **2º)** $\dfrac{t+1}{4} = \dfrac{t+2}{3}$ **3º)** $\dfrac{y-2}{7} = \dfrac{3}{y+2}$

2 **«If $\dfrac{a}{x} = \dfrac{x}{b}$, then x is the mean proportional between a and b».**

Calculate the mean proportional between a and b, in each of the following cases.

1º) $a = 1$ and $b = 9$

2º) $a = \dfrac{2}{3}$ and $b = \dfrac{50}{3}$

3º) $a = \sqrt{3}$ and $b = \sqrt{27}$.

3 Prove that the perimeter of a square is directly proportional to the measure of its side .

4 In each of the following cases, calculate the fourth proportional of the given numbers.

1º) 7 ; 8 and 3 **2º)** $\dfrac{1}{3}$; $\dfrac{1}{4}$ and $\dfrac{1}{5}$.

5 Divide 27 into parts directly proportional to 2; 3 and 4.

6 The magnitudes x and y being inversely proportional, complete the following table.

x	2	6	12	4
y	18			9

7 Divide 39 into parts inversely proportional to 2; 3 and 4.

8 Divide 310 into parts inversely proportional to 5 ; 7 and 10.

9 Answer by true or false.

1º) If the magnitudes x and y are inversely proportional, then the product $x \times y$ remains constant.

2º) If $2x = 3y = 4z$, then x, y and z are inversely proportional to 2; 3 and 4.

3º) If $2x = 3y = 4z$, then x, y and z are directly proportional to $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$.

4º) If $2x = 3y$, then $\frac{x}{y} = \frac{2}{3}$.

5º) The length of a circle is inversely proportional to its radius.

6º) The area of a disc is directly proportional to the square of its radius and π is the proportionality coefficient.

7º) If you increase a price by 100%, you multiply it by 2.

For seeking

10 Divide 1584 into parts inversely proportional to 2 ; 5 ; 6 and 9.

11 Three numbers x, y and z are directly proportional to 3 ; 5 and 6.

Complete : $\frac{x}{\dots} = \frac{\dots}{5} = \frac{\dots}{\dots} = k$.

Express x, y and z using k.

Knowing that : $2x + 3y + \frac{z}{2} = 48$.

Use this relation to calculate the numerical value of k.

Deduce the numerical values of x, y and z.

12 Three numbers x, y and z are inversely proportional to 12 ; 6 and 4.

Complete : $\frac{x}{\dots} = \frac{\dots}{\dots} = \frac{\dots}{\frac{1}{4}} = k$.

Express x, y and z using k.

Knowing that $2x - y + 3z = 9$.

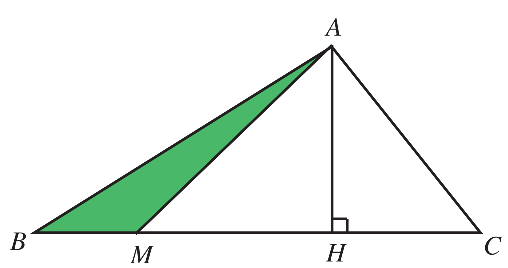
Use this relation to calculate the numerical value of k.

Deduce the numerical values of x, y and z.

13 $BC = 7$

$AH = 5$

$BM = x$



1º) Express the area $\mathscr{A}$ of the triangle ABM in terms of x .

2º) Are the magnitudes $\mathscr{A}$ and x directly proportional ? If yes, find the proportionality coefficient.

14 The demand of jeans on the market is inversely proportional to its purchase price.

If 1 500 jeans are sold when the purchase price is \$ 40, what would be the number of sold jeans if the purchase price becomes \$ 60 ?

15 Working 8 hours per day, a bricklayer would build a wall in 19 days.

If he wants to finish it in 16 days, how many hours per day should he work ?

TEST

1 The magnitudes x and y are inversely proportional.

Complete the following table.

x	8	$\frac{1}{4}$		0.5	
y	2		$\frac{2}{3}$		0.5

(3 points)

2 Calculate the number x, in each of the following cases :

1º) x is the fourth proportional of 3; 4 and 6. **(1 point)**

2º) x is the mean proportional between 3 and 12. **(1 point)**

3º) $\frac{x-1}{2} = \frac{x}{3}$. **(2 points)**

3 Divide 282 into parts inversely proportional to 5; 7 and 8.

(3 points)

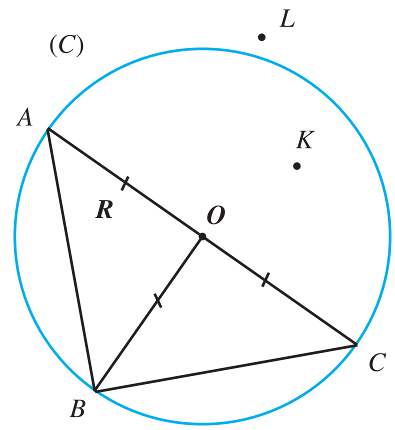
17 THE CIRCLE

Objectives

1. To know the relative positions of a straight line and a circle.
2. To determine the centers of the circles passing through two points and through three non-collinear points.
3. To recognize and to calculate the length of an arc of a circle.

CHAPTER PLAN

1 GENERALITIES



⊙ The blue line above represents a **circle** (C) of center O and of radius r : the line is made of points whose distance to O equals the constant r.

A is a point on (C) : $OA = r$.

K is a point inside (C) : $OK < r$.

L is a point outside (C) : $OL > r$.

⊙ ***O*** the center of (C) is a **center of symmetry** of (C).

⊙ A segment from the center O to a point on (C) is called the **radius** : $[OA]$ and $[OB]$ are two radii of (C).

⊙ A segment, having two points as extremities on (C), is called the **chord** : $[AB]$ and $[BC]$ are two chords.

$[AC]$ is a chord that passes through the center; it is called a **diameter**.

A and C are diametrically opposite

diameter = 2 × radius

⊙ Any diameter of (C) is an **axis of symmetry** of (C).

⊙ **The circumference** of $(C) = \mathbf{2\pi r}$.

$$\underset{3.14}{\pi} \qquad \underset{\text{radius of } (C)}{r}$$

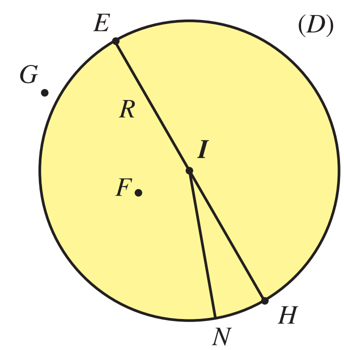
3.14 radius of (C)

⊙ The colored part shows a **disc** (D) of center I and of radius R : It is made of points whose distance to I is equal to or less than the constant R.

E is a point of (D) and $IE = R$.

F is a point of (D) and $IF < R$.

G is a point outside (D) and $IG > R$.



⊙ The center $\boldsymbol{I}$ of (D) is a **center of symmetry** of (D).

⊙ A segment, from the center I to a point E on (D) and such that $IE = R$, is called the **radius** of (D) : $[IE]$ and $[IN]$ are two radii of (D).

⊙ If E, I and H are three collinear points of (D) with $IE = IH = R$, then $[EH]$ is a diameter of (D).

diameter = 2 × radius

⊙ Any diameter of (D) is an **axis of symmetry** of (D).

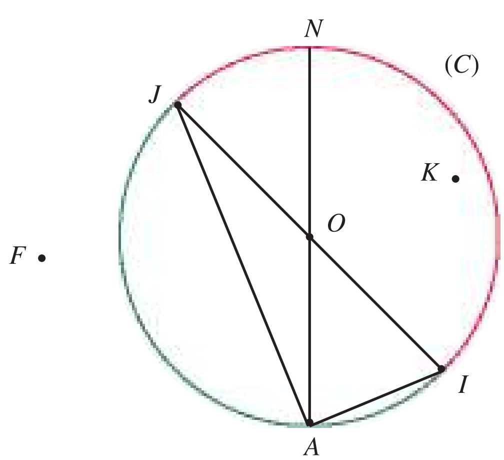
⊙ **Circumference** of (D) = $\boldsymbol{2\,\pi\,R}$.

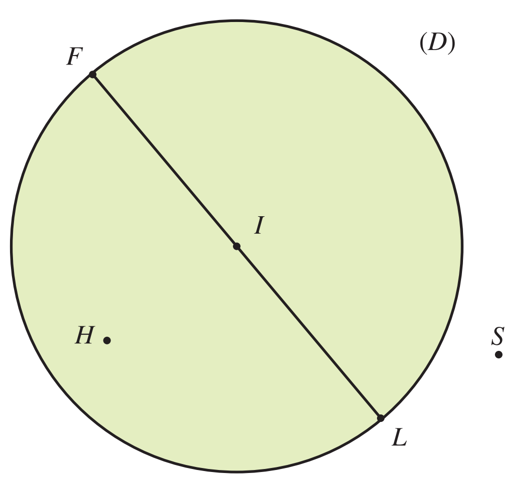
$\pi \to$ 3.14 $\quad R \to$ radius

⊙ **Area** of (D) = $\boldsymbol{\pi\,R^2}$ **.**

Application 1

Observe the circle (C) and the disc (D), then answer the following questions.





1°) Name two radii, a diameter, two chords, a point inside (C) and a point outside (C). Is O a point of (C) ?

2°) Name a radius and a diameter of (D). Is I a point of (D) ?

3°) What is the symmetric of A with respect to O ?

4°) Name four points of (D).

5°) Is the yellow semicircle congruent to the red semicircle ? Why ?

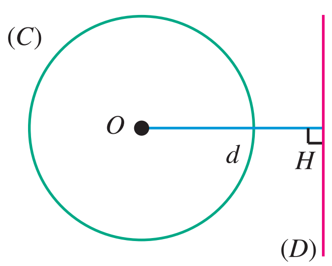
6°) Use a protractor to measure $\widehat{IAJ}$.

7°) Calculate the circumference of (C) if $IJ = 0.5$ cm.

8°) Calculate the circumference and the area of (D) when $LF = 10$ cm.

2 RELATIVE POSITIONS OF A LINE AND OF A CIRCLE

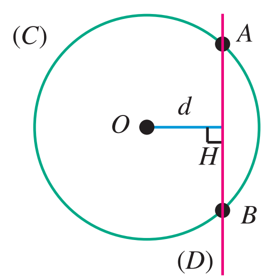
(C) is a circle of center O and of radius r. Let d be the distance from O to (D) : $d = OH$.



$d = OH > r$

(D) and (C) do not have any point in common.

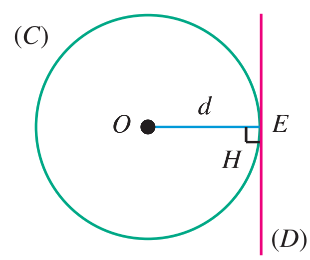
(D) is outside (C).



$d = OH < r$

(D) and (C) are secant. They intersect at A and B.

(D) is a secant to (C).



$d = OH = r$

(D) and (C) are tangent. They have one point in common E.

(D) is the tangent to (C) at E.

Application 2

1º) Draw a line (D) and a point O whose distance to (D) is 4 cm.

2º) Draw the circles $C\,(O\,,\,4\text{ cm})$ and $C'\,(O\,,\,5\text{ cm})$.

3º) What can you say about (C') and (D) ? about (C) and (D) ? Justify your answers.

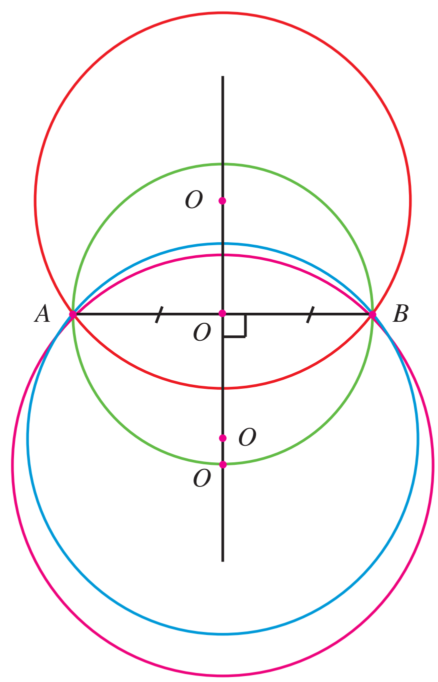
3 CIRCLES PASSING THROUGH TWO GIVEN POINTS

Property

A and B are two given points.

If O is the center of a circle that passes through A and B, then $OA = OB$, which means that O is equidistant from A and B.

Thus, O is on the perpendicular bisector of $[AB]$.



An **infinite** number of circles can pass through two given points ***A*** and ***B***. Their **centers** are on the **perpendicular bisector of [*AB*]**.

Application 3

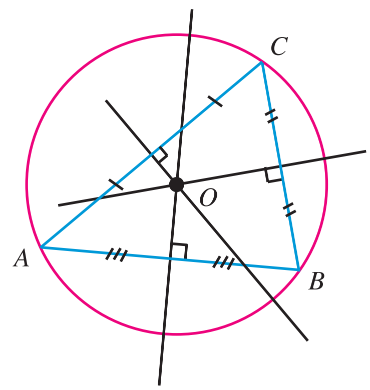
$[AB]$ is a segment measuring 4 cm.

1º) Draw the circle (C) of diameter $[AB]$.

2º) Draw two other circles passing through A and B.

4 CIRCLE PASSING THROUGH THREE GIVEN POINTS

Properties

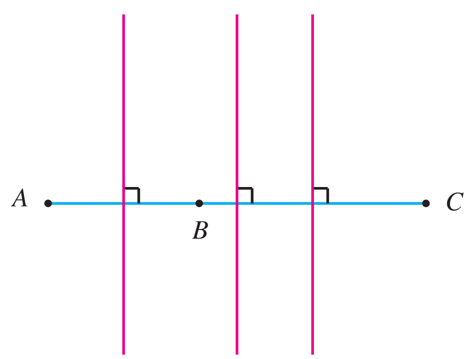


⊙ If O is the center of a circle passing through A, B and C, then $OA = OB = OC$. Being equidistant from A, B and C, O is the intersection point of the perpendicular bisectors of $[AB]$, of $[AC]$ and of $[BC]$; this point is unique.

One circle passes through three given non-collinear points.

Its **center** is the **intersection** of the perpendicular bisectors segments formed by these three points (It is called the circumcenter of the triangle).

⊙ If A, B and C are collinear, then the perpendicular bisectors of $[AB]$, $[AC]$ and of $[BC]$ would be parallel and O would not exist.



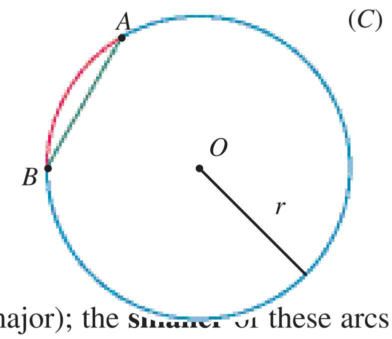
A circle does not pass through three **collinear** points.

Application 4

1º) Construct a triangle ABC, knowing that $AB = 4$, $AC = 3$ and $BC = 5$.

2º) Draw a circle passing through A, B and C. What does the center of this circle represent for $[BC]$?

5 ARC OF A CIRCLE



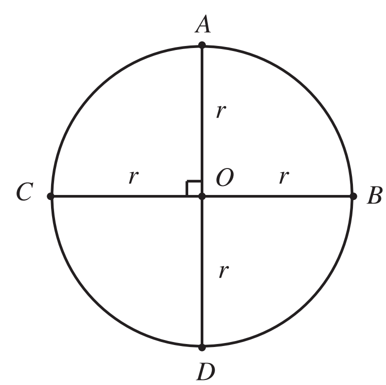
(C) is a circle of center O and radius r. A and B are two points on (C).

The part of (C) limited by A and B, is called **arc of a circle** with extremities A and B.

Therefore, A and B determine two arcs on the circle (minor and major); the **smaller** of these arcs is denoted by $\overset{\frown}{AB}$.

We say that the **chord [AB] is the corresponding chord of arc $\overset{\frown}{AB}$** .

Remark



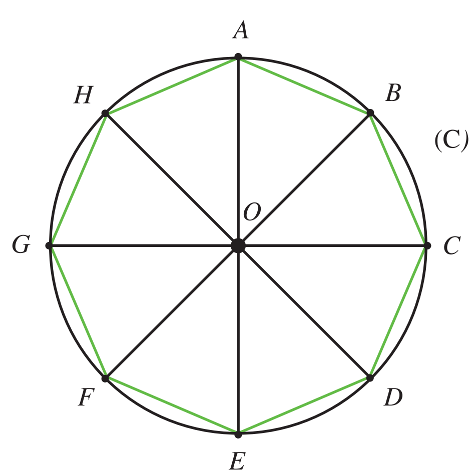
The arc of a circle has a length.

The circumference of the circle (C) is $2\,\pi\, r$.

The length of $\overset{\frown}{AB}$ is $\dfrac{2\,\pi\, r}{4} = \dfrac{\pi\, r}{2}$.

The length of $\overset{\frown}{BC}$ is $\dfrac{2\,\pi\, r}{2} = \pi\, r$.

Application 5

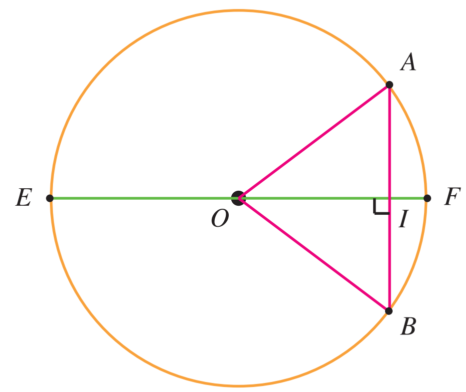


Observe the figure and answer.

1°) Name four arcs of (C) and the chords subtending them.

2°) Do the arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DE}$, $\overset{\frown}{EF}$, $\overset{\frown}{FG}$, $\overset{\frown}{GH}$ and $\overset{\frown}{HA}$ have the same length ? Calculate this length and the length of arc $\overset{\frown}{AF}$.

6 PROPERTY



In a circle, any diameter perpendicular to a chord, passes through the midpoint of this chord and of its arc.

<u>Given</u>	**<u>R.T.P.</u>**
$[EF]$ is a diameter	I is the
$[AB]$ is a chord	midpoint of
$[EF] \perp [AB]$ in I	$[AB]$

Proof

Join O to A and B.

$OA = OB$ (radius of a circle), therefore, the triangle OAB with vertex O is isosceles.

$[OI]$ is the height in this triangle.

Yet, in an isosceles triangle, the height relative to the base is the perpendicular bisector of this base. Therefore, I is the midpoint $[AB]$.

Application 6

(C) is a circle of center O; $[AB]$ is a chord of (C) with a midpoint I.

Prove that the diameter $[EF]$ that passes through I is perpendicular to $[AB]$.

Deduce the theorem.

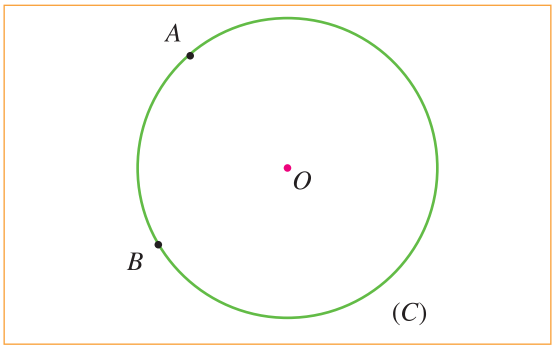
EXERCISES AND PROBLEMS

Test your knowledge

1 A and B are two points of a circle (C) of center O.

F is a point diametrically opposite to B . (AO) cuts again (C) at E.

Prove that $ABEF$ is a rectangle.



2 (C) is a circle of center O and of radius $r = 4$ cm.

$[AB]$ is a chord of (C) having a length of 4 cm.

1º) What is the nature of triangle OAB ? Justify.

2º) Color the minor arc subtended by chord $[AB]$.

3º) E is the symmetric of A with respect to O. Prove that E belongs to (C).

3 **1º)** On a line (d) place the points A, B, E, D such that :

$AB = 5$ cm, $BE = 3$ cm (E outside $[AB]$), $AD = 4$ cm (D is not between A and B).

2º) Draw $C\ (A, 5$ cm$)$, that is the circle (C) of center A and radius 5 cm.

3º) Draw the perpendiculars to (d) at B, E and D. Study the position of each of these perpendiculars with respect to (C).

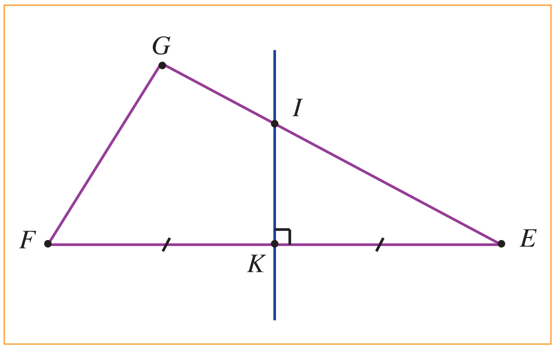
4 Construct a triangle EFG and its circumscribed circle in each of the following cases :

1º) $EF = 5$ cm , $EG = 8$ cm , $\widehat{FEG} = 110°$.

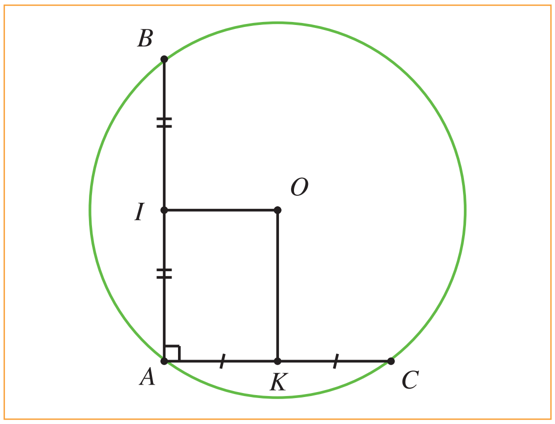
2º) $EF = 5$ cm , $EG = 8$ cm , $\widehat{FEG} = 70°$.

3º) $EF = 5$ cm , $EG = 8$ cm , $\widehat{FEG} = 90°$.

5 Show that $C(I, IE)$ passes through F.



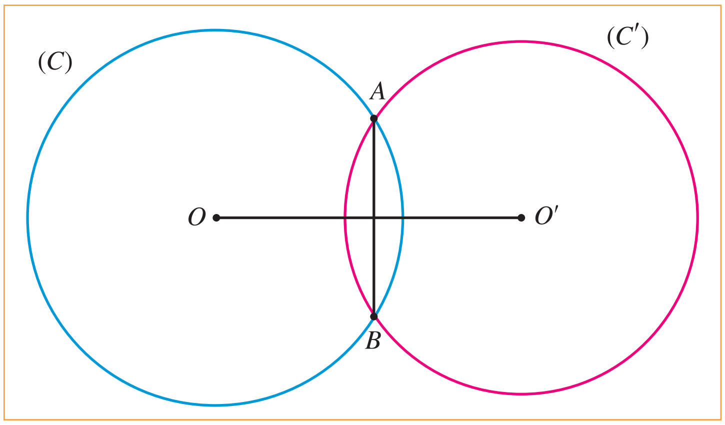
6 Observe the adjacent figure and show that quadrilateral $OIAK$ is a rectangle.



7 In the adjacent figure :

1º) Prove that (OO') is the perpendicular bisector of $[AB]$.

2º) What condition should be imposed on (C) and (C') so that quadrilateral $OAO'B$ would be a rhombus ?



8 Answer by true or false.

1º) If A, B and C are three collinear points, then the perpendicular bisectors of $[AB]$, $[AC]$ and $[BC]$ are parallel.

2º) The center of a circle circumscribed about a triangle (circle passing through the three vertices of a triangle) is not always inside this triangle.

3º) A circle can always pass through three given points.

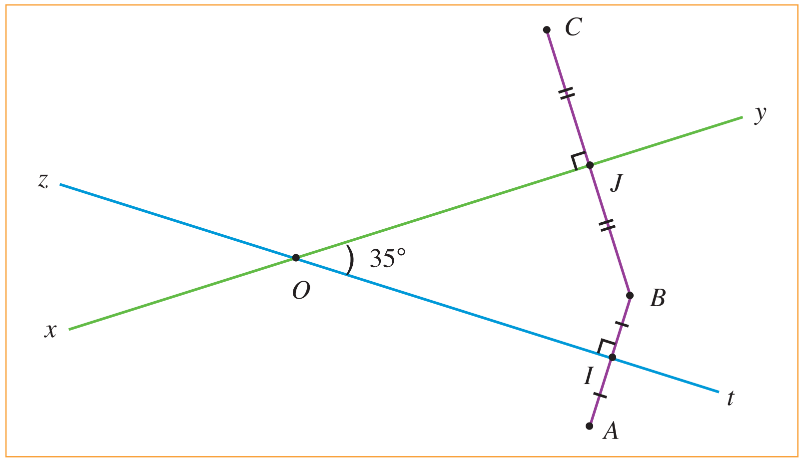
4º) The center of a circle is always the midpoint of any diameter of this circle.

5º) A circle has one axis of symmetry.

6º) In a circle of radius 8 cm, the length of the longest chord is 12 cm.

For seeking

9 In the adjacent figure,



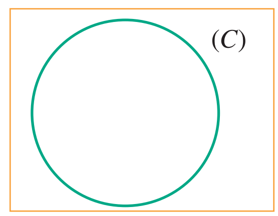
1°) Prove that the circle of center O and of radius OA passes through B and C.

2°) Prove that $\widehat{AOC} = 70°$.

10 **1°)** Construct a triangle ABC such that $AB = 6$ cm, $BC = 13$ cm and $AC = 7.5$ cm.

2°) Draw an arc of a circle passing through A, B and C.

11 Nabil has drawn a circle (C) and has lost the position of its center O. Help him to find O.

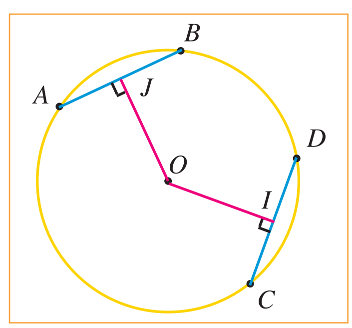


12 A and B are two given points.
1°) How many circles are there having for radius the segment $[AB]$?
2°) How many circles are there having for radius the length AB ?

13 Place two points E and F such that $EF = 6$ cm.
Color the part of the paper formed by the points that are less than 4 cm from E and less than 4 cm from F.

14 Using two strings F_1 and F_2 having the same length of 25.12 cm, make a circle (C) and a square (R).
Compare the areas of (R) and of the disc (D) limited by (C).

15 In the adjacent figure,



prove that :

1°) if $AB = CD$, then O is equidistant from the chords $[AB]$ and $[CD]$.

2°) if $OI = OJ$, then $AB = CD$.

TEST

1 **1º)** Draw a circle of center O and radius R. $[AB]$ is a chord of this circle.

2º) Show that the perpendicular bisector of $[AB]$ passes through O. **(1.5 point)**

3º) The perpendicular bisector of $[AB]$ cuts the circle at M and N. Name all the isosceles triangles of the figure and explain why they are isosceles. **(1.5 point)**

2 How many persons can stand up near each other around a circular pool, having a diameter of 36 m, if each person occupies 72 cm ?

(2 points)

3 The area of a disc is 530.66 cm². Calculate its diameter. **(2 points)**

4 Three points A, B, C divide a circle $C\,(O\,,\,90\text{ cm})$ into three arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ of respective lengths ℓ_1 , ℓ_2 , ℓ_3 .

Calculate ℓ_1 , ℓ_2 , ℓ_3 knowing that ℓ_1 is the double of ℓ_2 and that ℓ_3 is the triple of ℓ_1 .

(3 points)

18 RELATIVE POSITIONS OF TWO CIRCLES

Objective

To know the relative positions of two circles.

CHAPTER PLAN

COURSE

1. Line of centers
2. Exterior circles
3. Interior circles
4. Circles tangent externally
5. Circles tangent internally
6. Secant circles
7. Converse
8. Line of centers : axis of symmetry

EXERCISES AND PROBLEMS

TEST

COURSE

1 LINE OF CENTERS

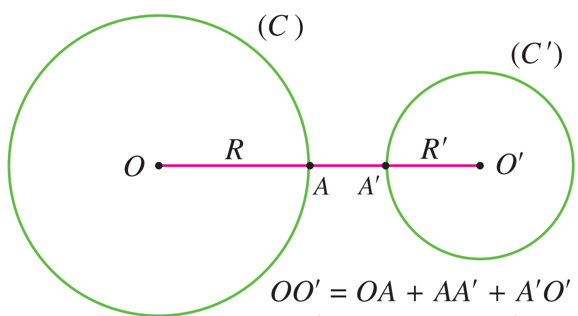
Consider a circle (C) of center O, and radius R . Consider another circle (C') of center O', and radius R'.

The line (OO') is called : line of centers.

2 EXTERIOR CIRCLES

These two circles do not have **any common point**, they are called **exterior**. The adjacent figure shows that :

$$\boldsymbol{OO' > R + R'}$$



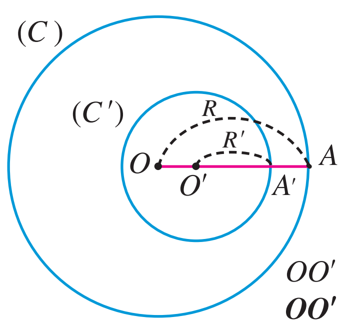
$OO' = OA + AA' + A'O'$

$\boldsymbol{OO' = R} + AA' + \boldsymbol{R'}$

3 INTERIOR CIRCLES

These two circles do not have **any common point**, they are called **interior**. The adjacent figure shows that :

$$\boldsymbol{OO' < R - R'}$$

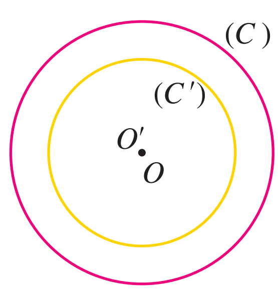


$OO' = OA - O'A' - A'A$

$\boldsymbol{OO' = R - R'} - \boldsymbol{A'A}$

Remark

If the centers O and O' **coincide**, the circles are called **concentric.**

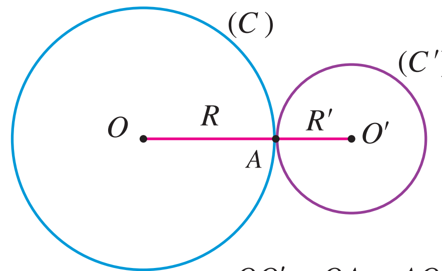


4 CIRCLES TANGENT EXTERNALLY

These two circles have **only one common point** A, they are called **externally tangent.**

The adjacent figure shows that :

$$\boxed{OO' = R + R'}$$



$OO' = OA + AO'$

$\mathbf{OO' = R + R'}$

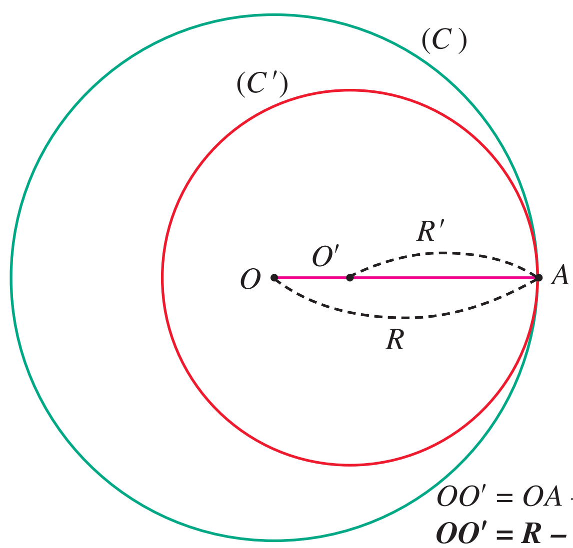
The common point A is called **point of tangency**; it is on the line of centers (OO').

5 CIRCLES TANGENT INTERNALLY

These two circles have **only one common point** A, they are called **internally tangent.**

The adjacent figure shows that :

$$\boxed{OO' = R - R'}$$

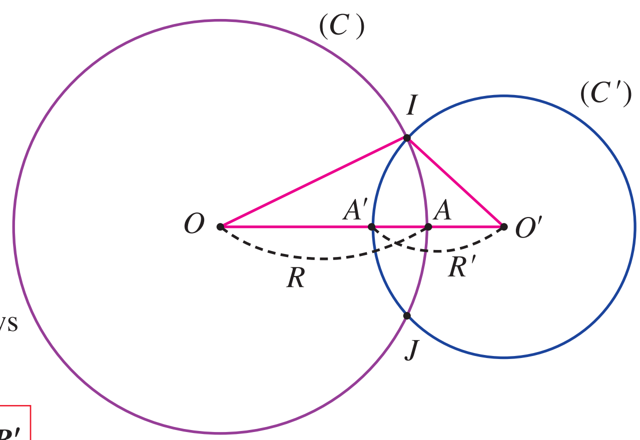


$OO' = OA - O'A$

$\mathbf{OO' = R - R'}$

A is the point of tangency; it is on line (OO').

6 SECANT CIRCLES



These two circles have **two common points** ***I* and *J***, they are called **secant.**

The adjacent figure shows that :

$$\boxed{\boldsymbol{R - R' < OO' < R + R'}}$$

OO' is the shortest distance between the two points O and O'.

$OO' < OI + O'I$ and $OI < OO' + O'I$

$OO' < R + R'$ and $OO' > R - R'$.

Therefore $\boldsymbol{R - R' < OO' < R + R'}$

Segment **[*IJ*]** is called the **common chord** of (C) and (C').

Remark

$OI = OJ = R$ and $O'I = O'J = R'$.
Therefore, (OO') is the perpendicular bisector of $[IJ]$.

7 CONVERSE

(C) and (C') are two circles of centers O and O', and radii R and R', respectively.

- **If $OO' > R + R'$, then (C) and (C') are exterior circles**
- **If $OO' < R - R'$, then (C) and (C') are interior circles (with $R > R'$)**
- **If $OO' = R + R'$, then (C) and (C') are externally tangent circles**
- **If $OO' = R - R'$, then (C) and (C') are internally tangent circles (with $R > R'$)**
- **If $R - R' < OO' < R + R'$, then (C) and (C') are secant circles (with $R > R'$)**

Application

$C(O , R)$ and $C'(O' , R')$ are circles, such that $R = 6$ cm and $R' = 4$ cm.
Without drawing the figure, study the position of both circles in each of the following cases :

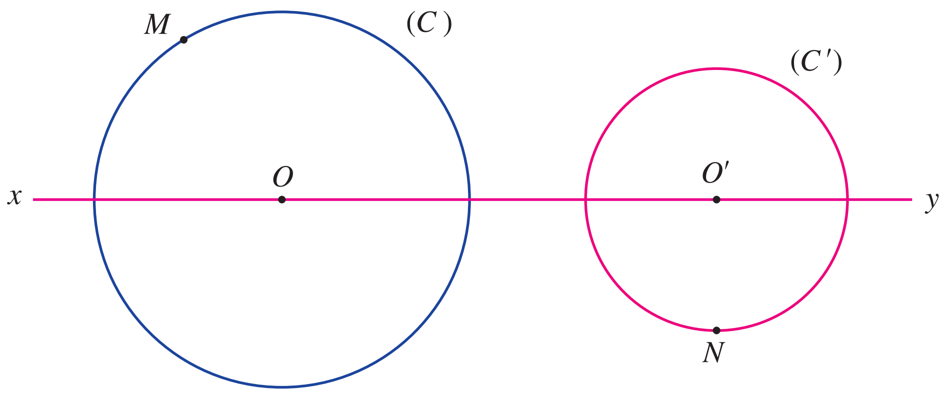
1°) $OO' = 12$ cm. **2°)** $OO' = 10$ cm. **3°)** $OO' = 3$ cm.
4°) $OO' = 1$ cm. **5°)** $OO' = 2$ cm.

8 LINE OF CENTERS : AXIS OF SYMMETRY

Activity

(C) and (C') are circles of centers O and O'; (xy) is the support of segment $[OO']$.

1°) M is a point on (C).



a) Construct point M' the symmetric of M with respect to (xy). What does (xy) represent for segment $[MM']$?

b) Where is M' located ?

2°) N is a point on (C').

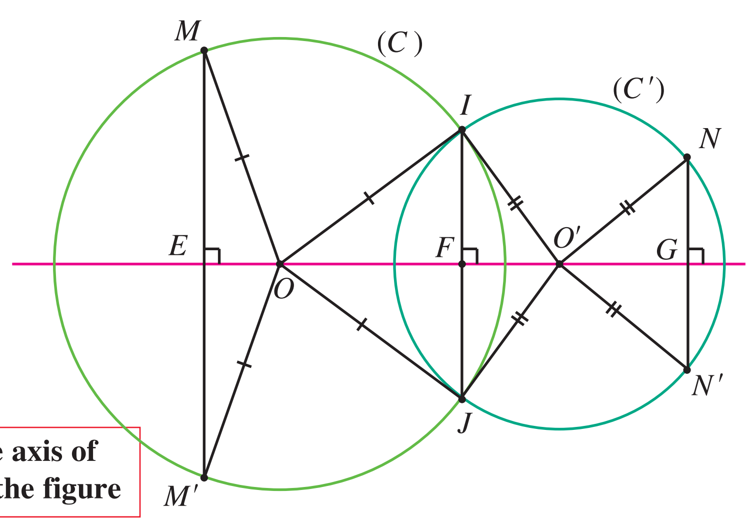
a) Construct point N' the symmetric of N with respect to (xy). What does (xy) represent for the segment $[NN']$?

b) Where is N' located ?

3°) Can you deduce that line (xy) is an axis of symmetry of this figure ?

Property

The line of centers (OO') is the perpendicular bisector of $[MM']$, $[IJ]$ and $[NN']$ with M and N on (C) and (C').



(OO') is the axis of symmetry in the figure

EXERCISES AND PROBLEMS

Test your knowledge

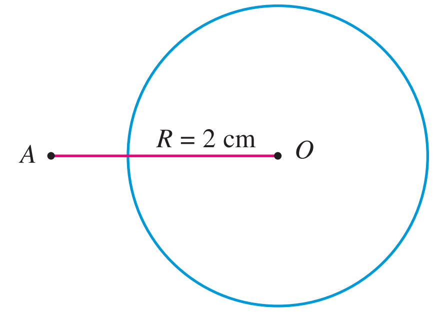
1 $C(O, R)$ and $C'(O', R')$ are two given circles with R and R' expressed in cm.

Complete the following table.

R	R'	OO'	Relation between OO', $R - R'$ and $R + R'$	Position of (C) and (C')
3	4	8		
2	5			Tangent externally
8		3	$OO' = R - R'$	
9	4	7		
10	6	2		
5.2		9	$OO' = R + R'$	

2 (C) is a circle of center O and radius $R = 2$ cm, and A is a point such that $OA = 3$ cm. There exist two circles of center A and tangent to (C).

Construct these circles; calculate their radii.



3 **1º)** Construct a circle C (0,3 cm). Let A be any point on (C).

2º) Construct the circles (C_1) and (C_2) having the same radius 1 cm and tangent at A to (C).

3º) What is the position of (C_1) and (C_2) ?

4 Answer by true or false.

1°) Two externally tangent circles or internally have one common point.

2°) Two circles, having no common point, are secant.

3°) Two interior circles do not have any common point.

4°) Two circles, having the same center and different radii, are concentric.

5°) The line joining the centers of two circles is their common chord.

6°) (C) and (C') are two circles of centers O and O', and radii R and R', respectively, such that $R = 8$ cm and $R' = 4$ cm.

a) If $OO' = 3$ cm, then (C) and (C') are exterior.

b) If $OO' = 12$ cm, then (C) and (C') are tangent internally.

c) If $OO' = 10$ cm, then (C) and (C') are secant.

d) If (C) and (C') intersect in A and B, then (AB) is the perpendicular bisector of $[OO']$.

e) (OO') is the axis of symmetry in this figure.

For seeking

5 (C_1) is a circle of center O_1 and radius $R_1 = 6$ cm; (C_2) is another circle of center O_2 and radius R_2 with $O_1O_2 = 10$ cm.

How should R_2 be chosen so that :

1°) (C_1) and (C_2) would be exterior ?

2°) (C_1) and (C_2) would be tangent externally ?

6 (C_1) is a circle of center O_1 and radius $R_1 = 8$ cm; (C_2) is another circle of center O_2 and radius R_2 with $O_1O_2 = 5$ cm.

How should R_2 be chosen so that :

1°) (C_1) and (C_2) would be interior ?

2°) (C_1) and (C_2) would be tangent internally ?

7 ABC is a right triangle at A. Draw a circle (C_1) of center B and radius BA; draw a circle (C_2) of center C and radius CA.

1°) What is the position of (C_1) and (C_2) ?

2°) Let D be the second point of intersection of (C_1) and (C_2).

a) Show that triangles ABC and BDC are congruent.

b) What is the nature of triangle BDC ?

8 ABC is an equilateral triangle of side 6 cm.

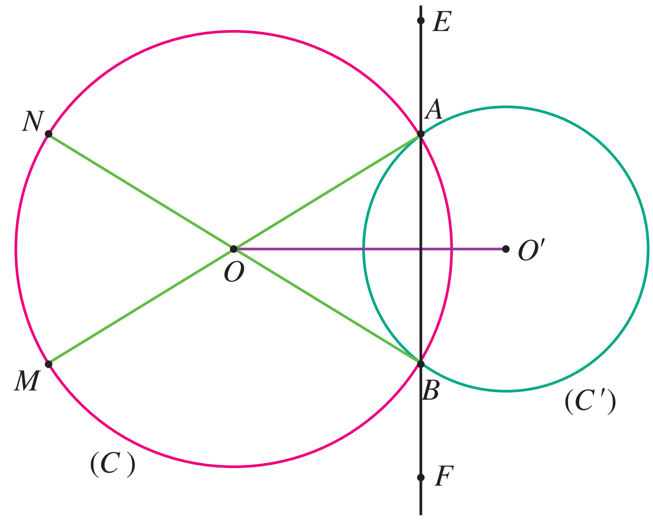
1°) Locate the center O of the circle (C) circumscribed about the triangle ABC . Construct (C).

2°) Prove that O is equidistant from (AB), (AC) and (BC).

9 Consider (C) a circle of center O. A is an interior point of (C) and B is a point on (C). (C') is a circle passing through A and B and of center O' on $[OB]$.

Precise the position of O' and of (C) with respect to (C').

10 $C\,(O)$ and $C'\,(O')$ intersect at A and B . Line (OA) cuts (C) again at M , and line (OB) cuts (C) again at N.



1°) Show that quadrilateral $ABMN$ is a rectangle.
2°) Show that (OO') is the perpendicular bisector of $[MN]$.
3°) On line (AB), place the two points E and F such that $AE = BF$ (show figure).
Show that $ME = NF$.

TEST

1 Two circles (C) and (C') with respective centers O and O' and having the same radius R , intersect at A and B.
Show that quadrilateral $OAO'B$ is a rhombus. **(2 points)**

2 $C\,(O\,,R)$ and $C'\,(O'\,,R')$ are two given circles.
Complete the following table.

R	R'	OO'	Position of (C) and (C')
4	3	8	
4	3		tangent externally
4	3	5	
7		4	tangent internally
7	3	2	
5	3	0	

(6 points)

3 $C_1\,(O_1\,,R_1)$ and $C_2\,(O_2\,,R_2)$ are externally tangent circles such that $O_1O_2 = 12$ cm.
Calculate R_1 and R_2 when $R_1 = 3R_2$. **(2 points)**

19 ARCS AND ANGLES

Objectives

1. To know and to use the relation between the measure of a central angle of a circle and that of its intercepted arc.
2. To know and to use the relation between the measure of an inscribed angle in a circle and that of its intercepted arc.
3. To calculate the area of an angular sector.

CHAPTER PLAN

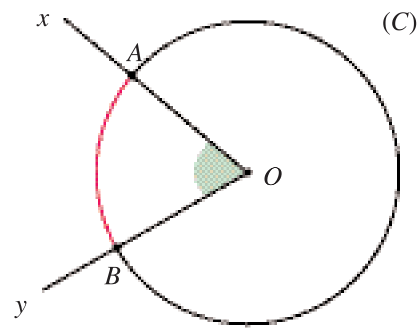
COURSE

EXERCISES AND PROBLEMS

TEST

1 CIRCLE AND ANGLES

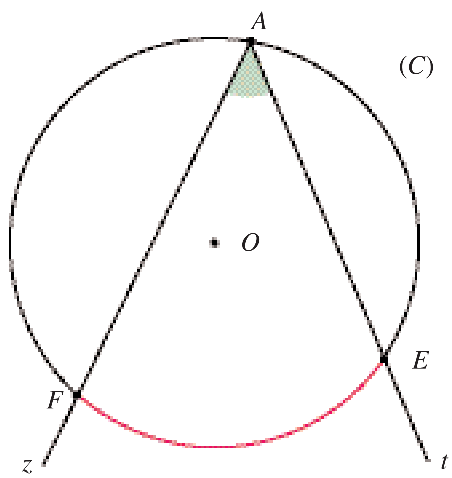
1º) Central angle



A **central angle** is an angle whose **vertex** is the **center of a circle**.

$\widehat{xOy}$ is a central angle; It intercepts arc $\overset{\frown}{AB}$.

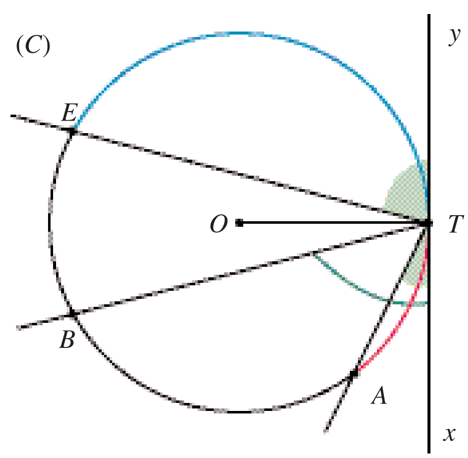
2º) Inscribed angle



An **inscribed angle** is an angle whose **vertex** is on the **circle** and whose sides are chords of the circle.

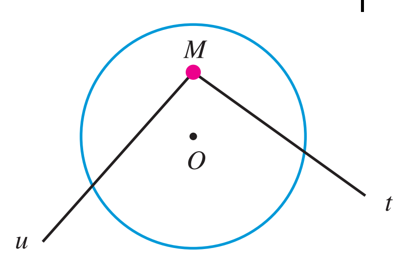
$\widehat{zAt}$ is an inscribed angle; it intercepts arc $\overset{\frown}{EF}$.

3º) Angle formed by a tangent and a chord intersecting at a point on the circle



An **angle**, formed by a **tangent** and a **chord** with the point of tangency being the vertex of the angle is considered an **inscribed angle**.
$\widehat{ATx}$, $\widehat{BTx}$ and $\widehat{ETy}$ are inscribed angles.

4º) Interior angle

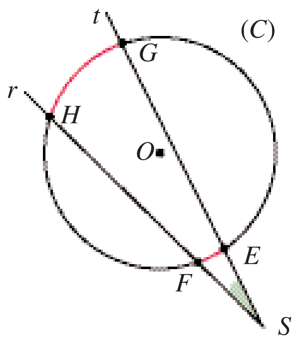


An **interior angle** is an angle having its **vertex inside the circle**.

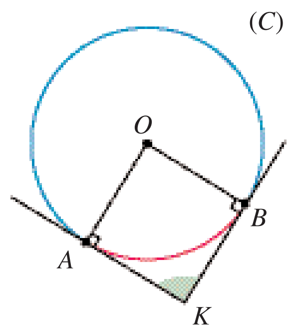
$\widehat{uMt}$ is an interior angle.

5º) Exterior angle

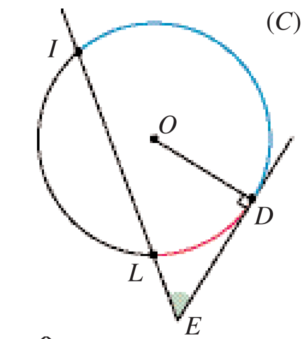
An **exterior angle** is an angle having its **vertex outside the circle** and its sides cut the circle.



$\widehat{rSt}$ is an exterior angle; it intercepts arcs $\overset{\frown}{EF}$ and $\overset{\frown}{GH}$.



$\widehat{BKA}$ is an exterior angle; it intercepts the arcs with extremities A and B.



$\widehat{DEL}$ is an exterior angle; it intercepts arcs $\overset{\frown}{DI}$ and $\overset{\frown}{DL}$.

Application 1

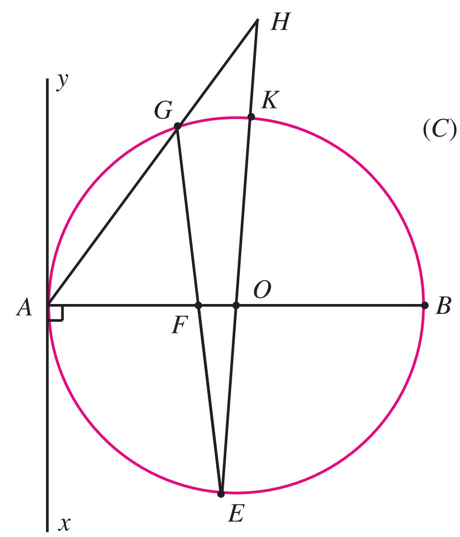
Observe the figure and name :

1º) two central angles

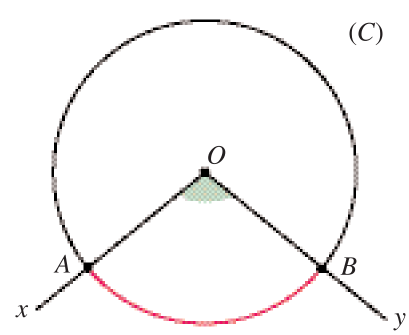
2º) six inscribed angles

3º) one interior angle

4º) one exterior angle.



2 CENTRAL ANGLE AND MEASURE OF AN ARC



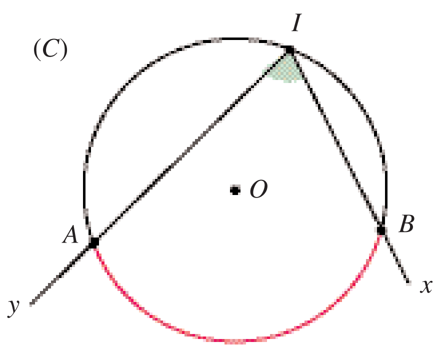
⊙ An **arc** $\overset{\frown}{AB}$ has a **measure** denoted by $\overset{\frown}{AB}$.

⊙ In a circle, **the central angle and the intercepted arc have the same measure.** This measure is expressed in degrees. For example,

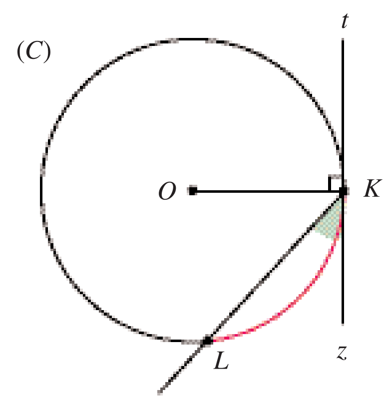
$$\widehat{xOy} = \overset{\frown}{AB}$$

3 MEASURE OF AN INSCRIBED ANGLE

In a circle, the measure of an inscribed angle is equal to half the measure of its intercepted arc.



$$\widehat{xIy} = \frac{\overset{\frown}{AB}}{2}$$

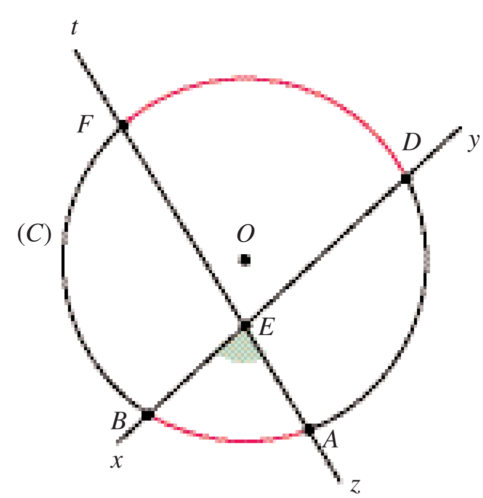


$$\widehat{LKz} = \frac{\overset{\frown}{LK}}{2}$$

4 MEASURE OF AN INTERIOR ANGLE

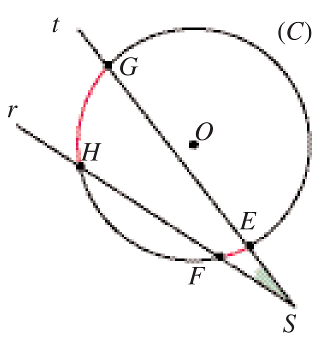
$$\widehat{yEt} = \widehat{xEz} = \frac{\overset{\frown}{AB} + \overset{\frown}{DF}}{2}$$

and $$\widehat{xEt} = \widehat{zEy} = \frac{\overset{\frown}{BF} + \overset{\frown}{AD}}{2}$$

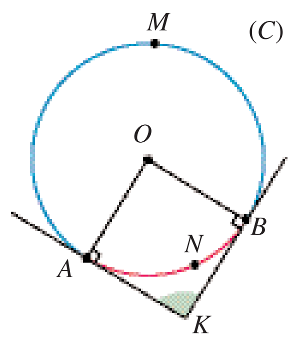


5 MEASURE OF AN EXTERIOR ANGLE

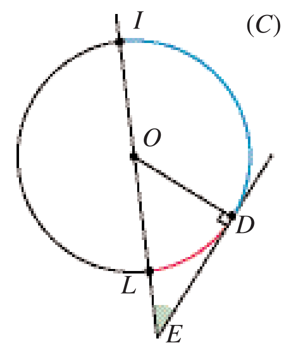
In a circle, an exterior angle is equal to half the difference of the intercepted arcs.



$$\widehat{tSr} = \frac{\overset{\frown}{HG} - \overset{\frown}{FE}}{2}$$



$$\widehat{BKA} = \frac{\overset{\frown}{AMB} - \overset{\frown}{ANB}}{2}$$



$$\widehat{IED} = \frac{\overset{\frown}{DI} - \overset{\frown}{LD}}{2}$$

6 SOLVED EXERCISES

Exercise 1

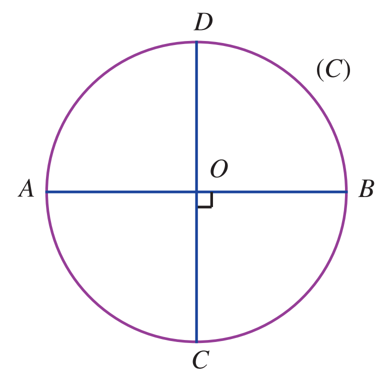
Let [AB] and [CD] be two perpendicular diameters of a circle C (O , 50 mm).

1°) Name the arc intercepted by each of the angles : $\widehat{DOB}$, $\widehat{BOC}$, $\widehat{COA}$ and $\widehat{AOD}$, Calculate the measure of each arc.

2°) Deduce the measure of the circle (C).

3°) Calculate the circumference of the circle (C).

Solution



1°) Angle $\widehat{BOD}$ intercepts arc $\overset{\frown}{BD}$, $\widehat{BOC}$ intercepts $\overset{\frown}{BC}$, $\widehat{COA}$ intercepts $\overset{\frown}{CA}$ and $\widehat{AOD}$ intercepts $\overset{\frown}{AD}$.

$\widehat{DOB} = 90°$ because $[AB] \perp [CD]$.

It is a central angle ; therefore

$\widehat{DOB} = \overset{\frown}{DB}$. Consequently : $\overset{\frown}{DB} = 90°$.

In the same way, we also prove that : $\overset{\frown}{BC} = \overset{\frown}{CA} = \overset{\frown}{AD} = 90°$.

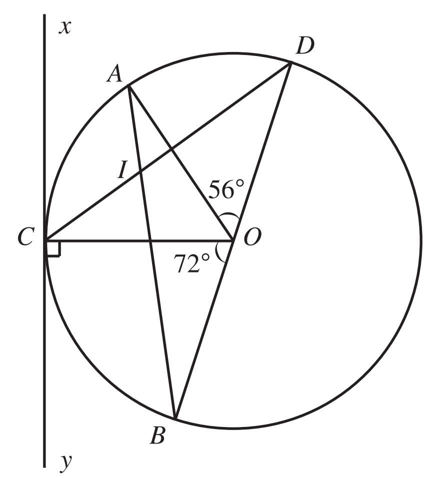
2°) The measure of (C) equals the sum of the measures of the arcs $\overset{\frown}{AC}$, $\overset{\frown}{CB}$, $\overset{\frown}{BD}$ and $\overset{\frown}{DA}$.
Therefore, the measure of : $90° + 90° + 90° + 90° = 360°$.

3°) The circumference of (C) is : $2 \times \pi \times r = 2 \times 3.14 \times 5 = 31.4$ cm.

Attention !

> Do not mix up between the circumference of a circle $(2 \times \pi \times r)$ and the measure of a circle $(360°)$.

Exercise 2



Observe the adjacent figure and calculate the angles :

$\widehat{AOC}$, $\widehat{CIB}$ and $\widehat{DCx}$.

Solution

Calculation of $\widehat{AOC}$

The points D, O and B are collinear.

Therefore $\widehat{DOB}$ is a straight angle :

$\widehat{DOA} + \widehat{AOC} + \widehat{COB} = 180°$,

$56° + \widehat{AOC} + 72° = 180°$,

$\widehat{AOC} = 180° - 56° - 72°$,

$\widehat{AOC} = 52°$.

Calculation of $\widehat{CIB}$

The angles $\widehat{DOA}$, $\widehat{AOC}$ and $\widehat{COB}$ are central angles; they respectively intercept the arcs $\overset{\frown}{DA}$, $\overset{\frown}{AC}$ and $\overset{\frown}{CB}$.

Yet, the central angle and the intercepted arc have the same measure.

Therefore $\overset{\frown}{DA} = 56°$, $\overset{\frown}{AC} = 52°$ and $\overset{\frown}{CB} = 72°$.

$\widehat{CIB}$ is an interior angle; therefore :

$$\widehat{CIB} = \frac{\overset{\frown}{CB} + \overset{\frown}{DA}}{2} = \frac{72° + 56°}{2} = 64° .$$

Calculation of $\widehat{DCx}$

$\widehat{DCx}$ is formed by the tangent to the circle at C and chord [CD]. It is considered an inscribed angle.

$$\widehat{DCx} = \frac{\overset{\frown}{DC}}{2} = \frac{\overset{\frown}{DA} + \overset{\frown}{AC}}{2} = \frac{56° + 52°}{2} = 54° .$$

Application 2

1º) Draw a circle (C) of center O and radius 4 cm.

2º) Consider [AB] a chord of (C) having a length of 4 cm. Calculate the measure and the length of arc $\overset{\frown}{AB}$.

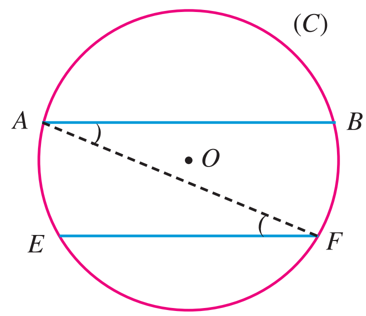
Note : Use the cross product property to calculate the length of arc $\overset{\frown}{AB}$.

7 PROPERTIES

1º) In a circle, **two parallel chords intercept between them two equal arcs.**

In the adjacent circle (C), the chords $[AB]$ and $[EF]$ are parallel; therefore :

$$\overset{\frown}{AE} = \overset{\frown}{BF}\,.$$

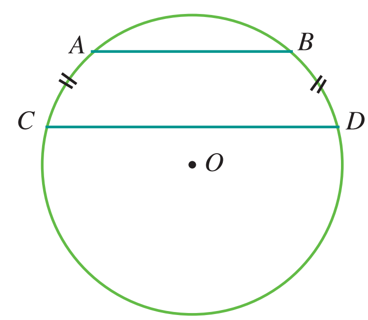


2º) In the adjacent circle (C), the chords $[AB]$ and $[CD]$ form between them two equal arcs :

$\overset{\frown}{AC} = \overset{\frown}{BD}$.

Therefore, these chords are parallel :

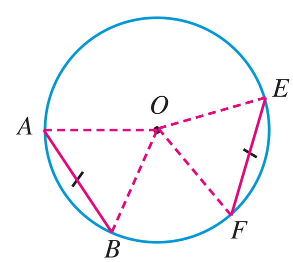
$(AB) \,//\, (CD)$.



3º) In a circle, **two congruent chords have their corresponding arcs congruent.**

In the adjacent circle (C), the chords $[AB]$ and $[EF]$ are congruent; therefore :

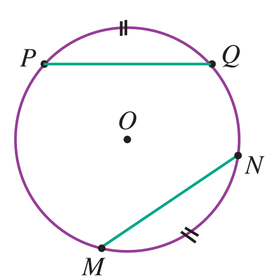
$$\overset{\frown}{AB} = \overset{\frown}{EF}\,.$$



4º) In a circle, **two equal arcs have their corresponding chords equal.**

In the adjacent circle (C), the arcs $\overset{\curvearrowright}{PQ}$ and $\overset{\curvearrowright}{MN}$ are equal; therefore :

$PQ = MN$.



8 RIGHT TRIANGLE AND CIRCLE

Activity

1º) Construct a triangle ABC such that $AB = 3$ cm, $AC = 4$ cm and $BC = 5$ cm.

2º) Use the pythagorean theorem to show that triangle ABC is right at A.

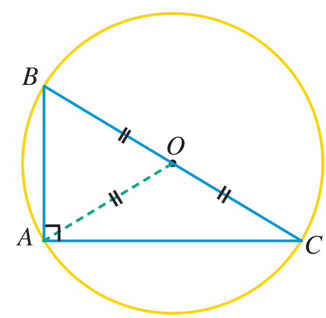
3º) Let I be the midpoint of $[BC]$; Prove that $IA = IB = IC$.

4º) Draw the circle (C) circumscribed about triangle ABC.

5º) What does $[BC]$ represent for circle (C) ? For triangle ABC ?

Property 1

If a right triangle is inscribed in a circle then its hypotenuse is a diameter of the circle.

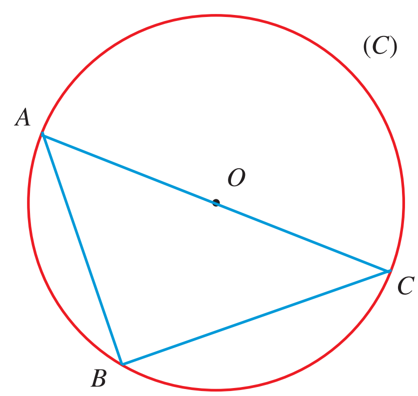


The adjacent triangle ABC is right at A; therefore, it is inscribed in the circle of diameter $[BC]$.

Property 2

If one side of an inscribed triangle is a diameter, then the triangle is right and the angle opposite to the diameter is right.

The adjacent triangle ABC is inscribed in the circle of diameter $[AC]$ which is one of its sides; therefore, ABC is right at B.



Application 3

«A quadrilateral is inscribed if its vertices are on the same circle».

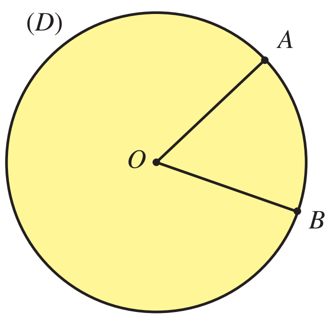
Prove that the rectangle *ABCD* is inscribed. Indicate the center of the circle on which these points are found.

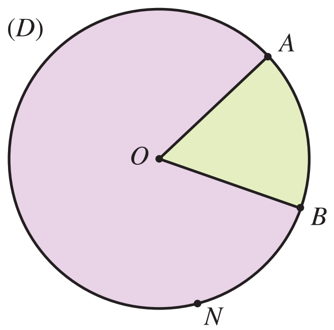
9 CIRCULAR SECTOR

1º) Definition of a circular sector

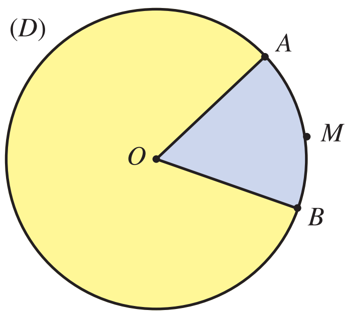
Consider (D) a disc limited by a circle (C) of center O and radius r.

A **circular sector** is a part of (D), bounded by two radii of the circle and their intercepted arc.



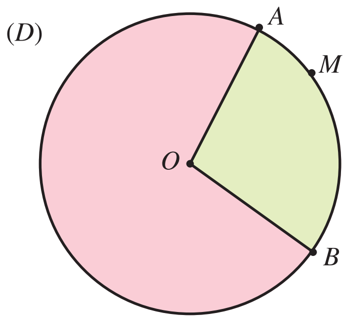


The circular sector *OANB* .



The circular sector *OAMB* .

2º) Area of a circular sector



Consider (D) a disc limited by the circle (C) of center O and radius r.

Consider two points A and B, on (C).

We calculate the area of the circular sector *OAMB*.

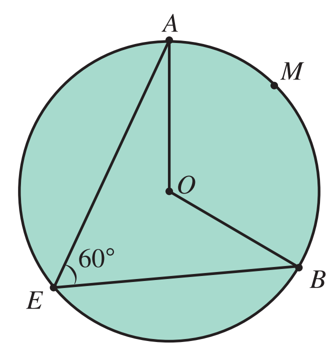
If α is the measure in degrees of the central angle $\widehat{AOB}$, then α is the measure of arc $\overset{\frown}{AB}$.

measure of (C)		area of (D)
to **360°**	we correspond	$\pi \times R^2$
to α	we correspond	x
area of $\overset{\frown}{AB}$		area of the circular sector $OAMB$

Hence : $x = \dfrac{\pi \times R^2 \times \alpha}{360°}$

Application 4

In the following figure, calculate the area of the circular sector *OAMB*.

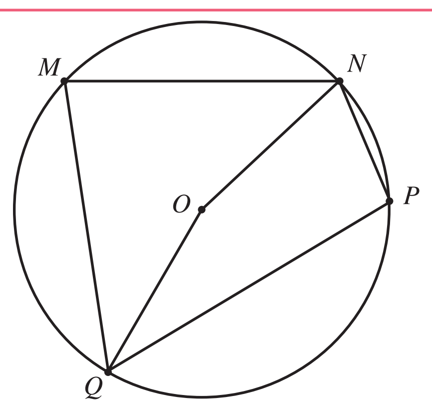


EXERCISES AND PROBLEMS

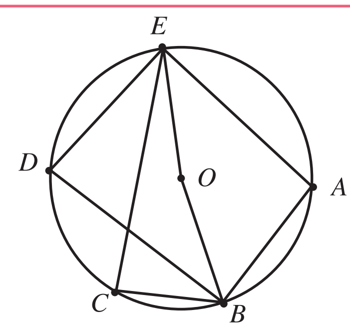
Test your knowledge

1 Name the angles that intercept the same arc.

1°)



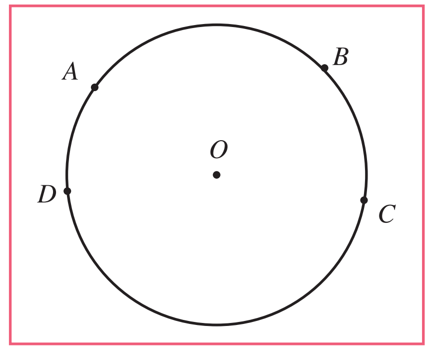
2°)



2 In the adjacent figure :

$\overset{\frown}{AB} = 100°$, $\overset{\frown}{BC} = 55°$ and $\overset{\frown}{DB} = 140°$.

Find : $\overset{\frown}{AD}$; $\overset{\frown}{AC}$; $\overset{\frown}{CD}$ and $\widehat{CAD}$.



3 What do we call :

1°) the segment that joins two points of a circle ?

2°) the line that has only one common point with the circle ?

3°) the chord that passes through the center of the circle ?

4°) a portion of the circle limited by two points on the circle ?

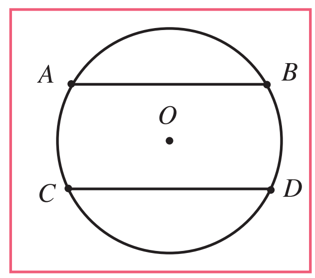
5°) the line that cuts the circle at two points ?

4 In the adjacent circle :

$(AB) // (CD)$, $\overset{\frown}{AC} = 55°$ and $\overset{\frown}{AB} = 105°$.

What is the measure of :

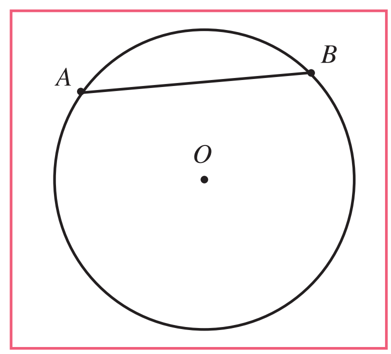
$\overset{\curvearrowright}{BD}$; $\overset{\curvearrowright}{AD}$; $\widehat{CAD}$ and $\overset{\curvearrowright}{CD}$?



5 **1°)** In the adjacent figure draw the perpendicular bisector of chord $[AB]$, which cuts the circle at X and Y (the point X is on the minor arc $\overset{\curvearrowright}{AB}$).

2°) What is segment $[XY]$ called ?

3°) If $\overset{\frown}{AB} = 80°$, find : $\overset{\frown}{AX}$; $\overset{\frown}{XY}$ and $\overset{\frown}{BY}$.

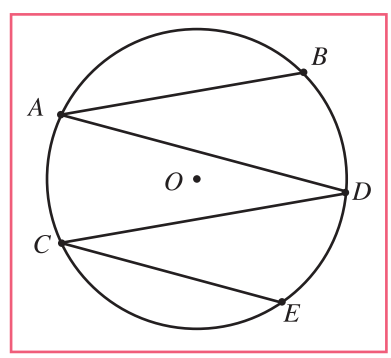


6 In the adjacent circle :

$(AB) // (CD)$ and $(AD) // (CE)$.

Knowing that $\overset{\frown}{AC} = 50°$ and $\overset{\frown}{CE} = 100°$, find :

$\overset{\frown}{BD}$; $\overset{\frown}{DE}$ and $\overset{\frown}{AB}$.

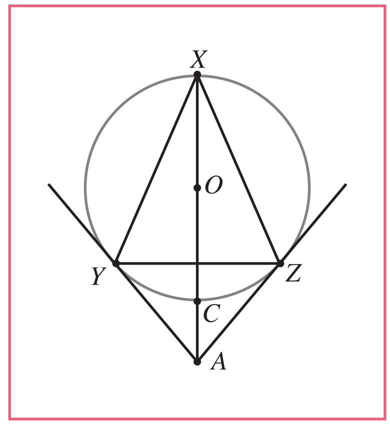


7 Consider triangle XYZ inscribed in a circle of center O.

$\widehat{XYZ} = \widehat{XZY} = 70°$,

$[XC]$ is a diameter of this circle; (AY) and (AZ) are tangents.

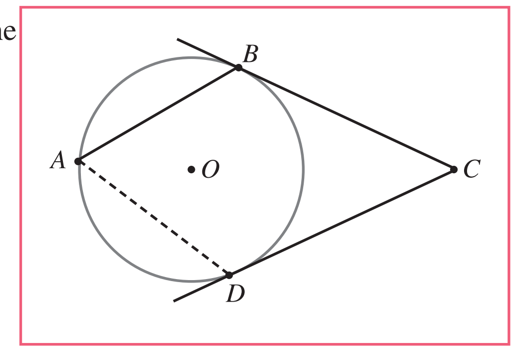
Find : $\widehat{YXZ}$; $\overset{\frown}{YZ}$; $\overset{\frown}{YX}$; $\overset{\frown}{YXZ}$; $\overset{\frown}{CZ}$; $\widehat{YAZ}$.



8 $\widehat{BCD}$, formed by two tangents, is an exterior angle to the circle of center O.

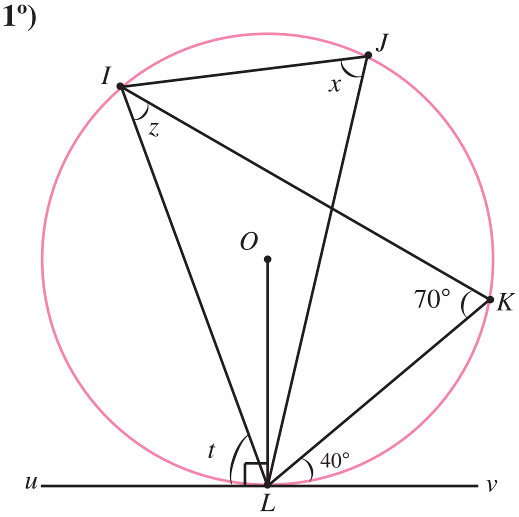
If $\overset{\frown}{AB} = 110°$ and $\overset{\frown}{BD} = 135°$, find :

$\overset{\frown}{AD}$; $\widehat{ABC}$; $\widehat{DCB}$ and $\widehat{BAD}$.

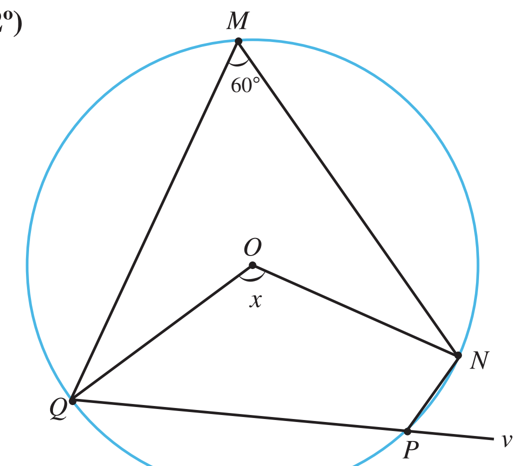


9 Calculate angles x, y, z and t.

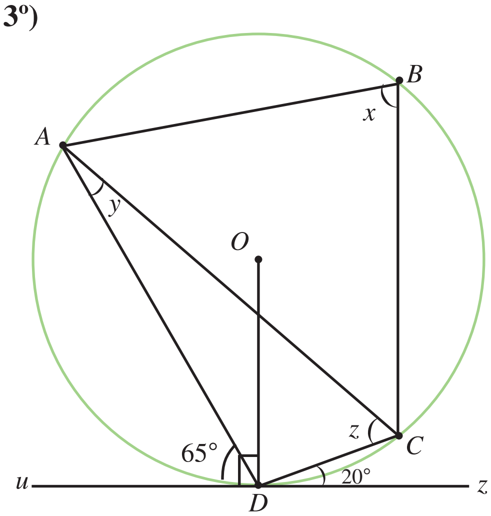
1º)



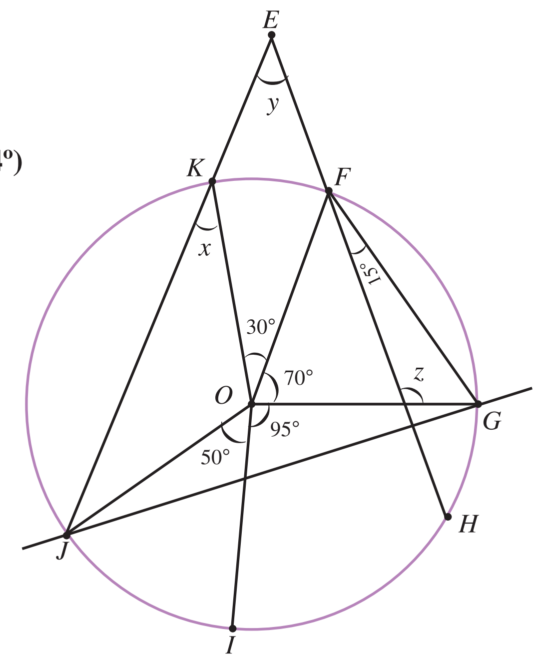
2º)



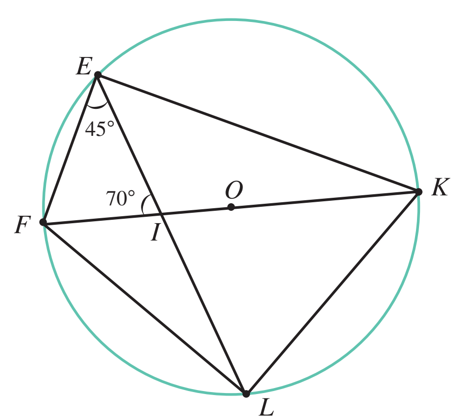
3º)



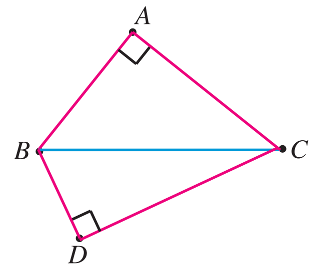
4º)



10 Observe the figure and calculate the angles of the quadrilateral *EFLK*.



11 Observe the figure and prove that the points A, B, C and D belong to the same circle.



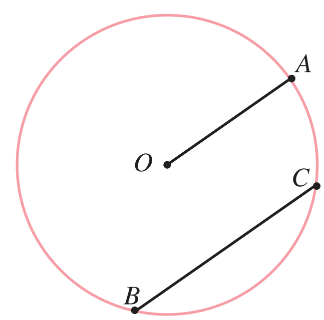
12 ABC is a triangle inscribed in a circle. Let D be the midpoint of arc $\overset{\frown}{AC}$. The parallel from D to (AB) cuts (BC) at I and the arc intercepted by $\widehat{BAC}$ at E.

1°) Compare the arcs $\overset{\frown}{BE}$ and $\overset{\frown}{DC}$.

2°) Show that : $IB = ID$ and $IE = IC$.

13 Let $[OA]$ be the radius of a circle of center O and $[BC]$ a chord parallel to (OA).

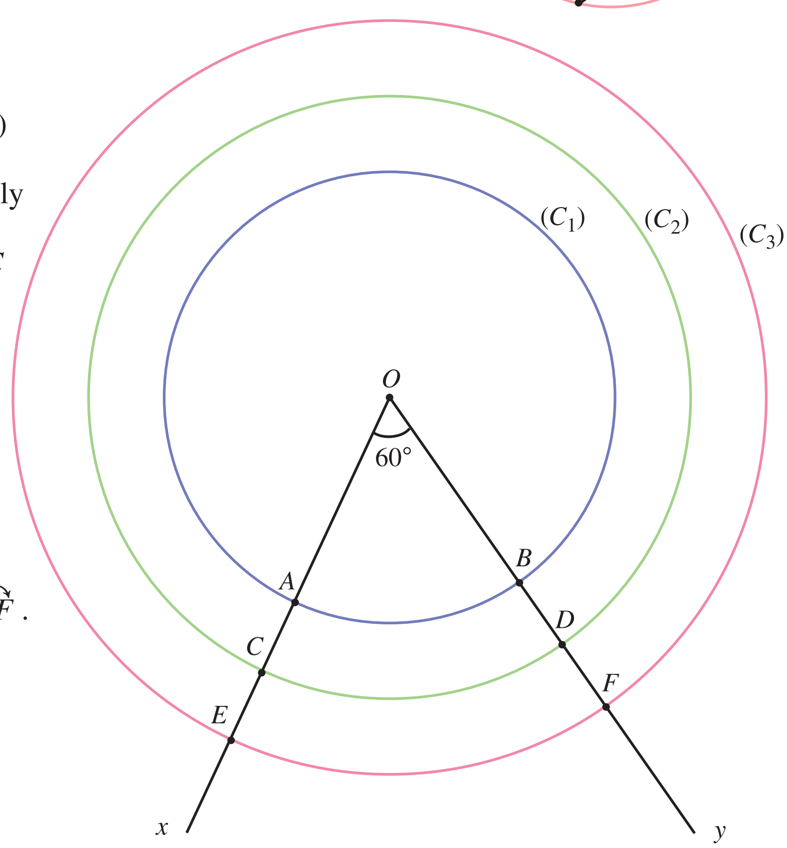
Prove that the difference of the measures of the angles $\widehat{B}$ and $\widehat{C}$ of triangle ABC is $90°$.



14 $\widehat{xOy} = 60°$.

$C_1\ (O, 3\text{ cm})$, $C_2\ (O, 4\text{ cm})$ and $C_3\ (O, 5\text{ cm})$ respectively cut $[Ox)$, $[Oy)$ at A and B, C and D, E and F (see the figure).

Calculate the measure and the length of each of the arcs $\overset{\frown}{AB}$, $\overset{\frown}{CD}$ and $\overset{\frown}{EF}$.



15 (C) is a circle of center O and radius 8 cm. A and B are two points on (C), such that $\widehat{AOB} = 60°$. The bisector of $\widehat{AOB}$ cuts (C) at K and L (K is between A and B).

1º) Calculate the circumference of (C).

2º) Calculate the length of arc $\overset{\frown}{AB}$.

3º) Calculate the area of the disc limited by (C).

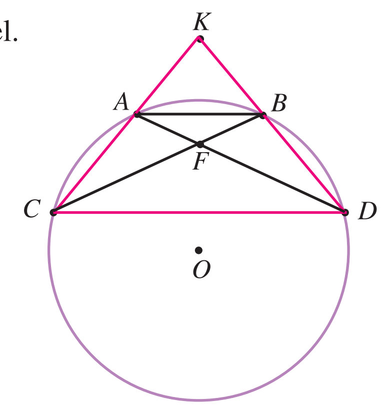
4º) Calculate the area of the circular sector $OAKB$.

5º) Calculate the angles $\widehat{ALB}$ and $\widehat{AKB}$.

16 In the following figure, the lines (AB) and (CD) are parallel.

1º) Prove that triangle AFB is isosceles.

2º) Prove that triangle KCD is isosceles.



17 Consider a triangle ABC inscribed in a circle (C) of center O and radius r. The supports of the altitudes $[AH]$ and $[BK]$ cut again the circle at D and E respectively.

Show that C is the midpoint of arc $\overset{\frown}{DE}$.

18 **1º)** In a circle (C) of center O and radius r, place the points E, F, G and H in the given order, such that : $\widehat{EOF} = 70°$, $\widehat{FOG} = 50°$, $\widehat{GOH} = 70°$.

2º) Calculate the angles of quadrilateral $EFGH$.

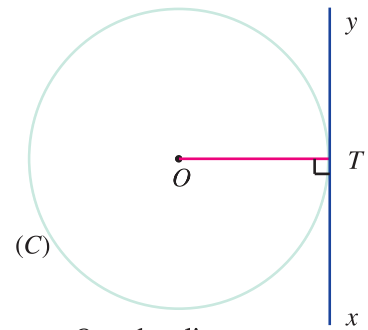
3º) What is the nature of quadrilateral $EFGH$? Justify.

19 Let A be an exterior point to a $C\ (O\ ,\ r)$. Two secants, passing through A, respectively cut (C) at B and E, and at C and D.

Show that if B is between A and E, and C is between A and D, then :

$\widehat{BAC} = \frac{1}{2}\left(\widehat{DOE} - \widehat{COB}\right)$.

20 (xy) is a tangent at T to a circle $C\ (O\ ,\ 3\text{cm})$. B and C are points of (C) such that $\widehat{xTB} = 55°$ and $\widehat{yTC} = 70°$.



1º) What is the nature of triangle TBC ? Justify.

2º) Calculate the area of the circular sector COT .

21 $[AB]$ and $[AC]$ are two congruent chords of a circle (C) of center O and radius r.

1º) Prove that $\overset{\frown}{AB} = \overset{\frown}{AC}$.

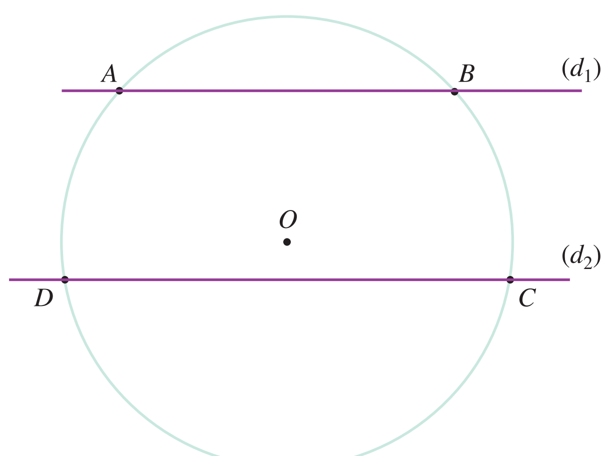
2º) A line (xy), passing through A, cuts (C) again at I and intersects (BC) at J (J is not on the segment $[BC]$).
Prove that $\widehat{ABI} = \widehat{AJB}$.

3º) Suppose that $\widehat{BAC} = 80°$; calculate $\overset{\frown}{AB}$.

22 Answer by true or false.
1º) In a circle, if two angles at the center are equal, then they intercept two equal arcs.
2º) In a circle, if an inscribed angle and a central angle intercept the same arc, then the first is twice the second.
3º) The measure of any circle is 360°.
4º) If A, B, C, D are four points such that $\widehat{BAC} = \widehat{BDC} = 90°$, then these points are on the circle of diameter $[BC]$.
5º) In a circle, two arcs having the same length have the same measure in degrees.

For seeking

23 (C) is a circle of center O and radius r.
(d_1) and (d_2) are two parallel lines that cut (C) at A, B, C and D.



1º) Show that quadrilateral $ABCD$ is an isosceles trapezoid.

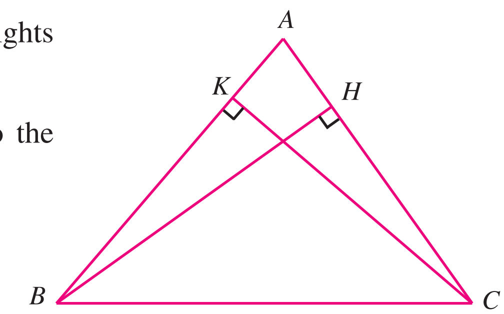
2º) Calculate the area of $ABCD$, knowing that $AB = 4$ cm, $CD = 10$ cm and the distance between (d_1) and (d_2) is 3 cm.

24 (C) is a circle of center O. (xy) is tangent to (C) at E . F and G are points of (C) such that $[EF)$ is the bisector of $\widehat{AEG}$ $\left(A \text{ is a point on } (xy)\right)$.

Prove that triangle EFG is isosceles.

25 ABC is a triangle. H and K are the feet of the heights drawn respectively from B and C .

Prove that the points K, H, B and C belong to the same circle.



26 Consider two secant circles $C\,(O\,,\,r)$ and $C'\,(O'\,,\,r)$ intersecting at A and B. (AO) cuts (C) at M and (BO') cuts (C') at M' .
What is the nature of quadrilateral $AM'BM$? Justify.

27 Consider H the orthocenter of a triangle ABC inscribed in a circle $C\,(O,r)$. The support of $[AH]$ cuts the circle (C) again at K.

1°) Show that $\widehat{CAH} = \widehat{CBK}$.

2°) Deduce that $\widehat{CBH} = \widehat{CBK}$ and that K is the symmetric of H with respect to (BC).

28 (C) is a semicircle of diameter $[AB]$, center O and radius r.
(xy) is tangent to (C) at E.

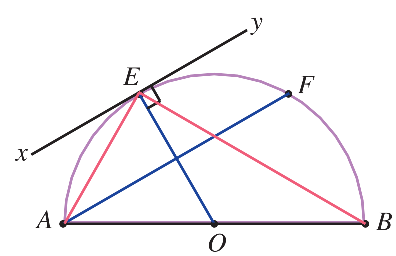
$(xy) \parallel (AF)$ and $\widehat{BAF} = 30°$.

1°) Calculate $\widehat{AEx}$ and $\widehat{BEy}$.

2°) Prove that AEB is a right triangle.

3°) Calculate the sides of triangle AEB when $r = 3$ cm.

4°) Calculate the area of the circular sector $OEFB$.



29 ABC is an equilateral triangle; O is the center of the circle circumscribed about this triangle. Let M be a point of the arc intercepted by angle A.

1°) Calculate $\widehat{BMA}$ and $\widehat{CMA}$. What does $[MA)$ represent for angle $\widehat{BMC}$?

2°) A ray $[Bz)$ cuts $[AM]$ at D such that $\widehat{DBM} = 60°$.

a) What is the nature of triangle BMD ?

b) Prove that triangles BDA and BMC are congruent.

3°) Prove that $MA = MB + MC$.

30 $C\,(O\,,r)$ and $C'\,(O'\,,r)$ are externally tangent circles at A. (TT') is a common tangent to (C) and (C') (see figure).

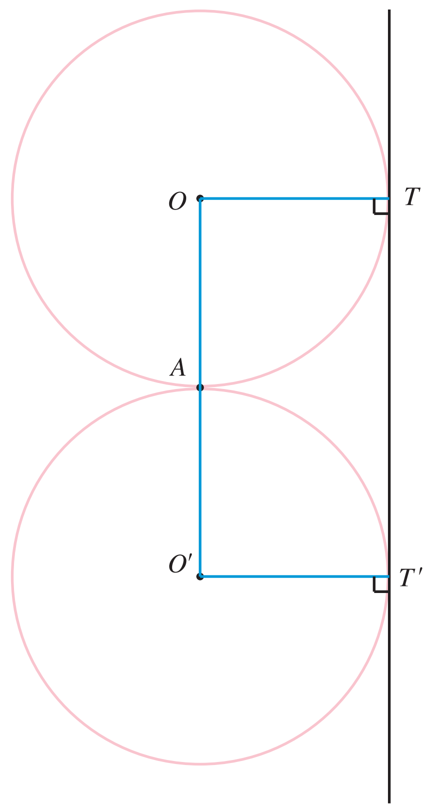
The perpendicular at A to (OO') cuts (TT') at H.

1°) Prove that quadrilateral $OTT'O'$ is a rectangle.

2°) What is the nature of quadrilateral $OTHA$? Justify.

3°) Prove that the vertices of $OTT'O'$ are on a circle (C_1); express the radius of this circle in terms of r.

4°) Prove that ATT' is a right triangle at A.



31 (C) is a circle of center O. $[AD]$ is chord of (C). The ray $[ON)$, perpendicular to (AD), cuts the circle at B $\big(N$ is a point of $[AD]\big)$. M is the orthogonal projection of O on (AB) and I is the intersection of (OM) and (AD).

1°) Show that (BI) is perpendicular to (OA).

2°) (BI) cuts the circle again at B' ; show that $AB' = AB$.

3°) Compare the triangles ADB and $AB'B$.

4°) Compare the triangles AIB' and BID.

5°) What is the nature of quadrilateral $ABDB'$? Justify.

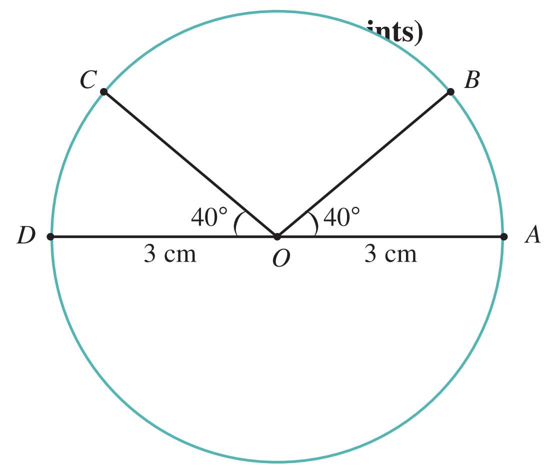
32 Two circles (C) and (C'), of respective centers O and O', intersect at A and B. Consider the points C and D, diametrically opposite to B on the circles (C) and (C') respectively.

1°) Show that the points C, A and D are collinear.

2°) Compare OO' and CD.

TEST

1 Observe the figure and answer the following questions. ints)



1º) Calculate angle $\widehat{BOC}$.

2º) Show that $\overset{\frown}{AB} = \overset{\frown}{CD}$.

3º) Find the measure of arc $\overset{\curvearrowright}{BC}$.

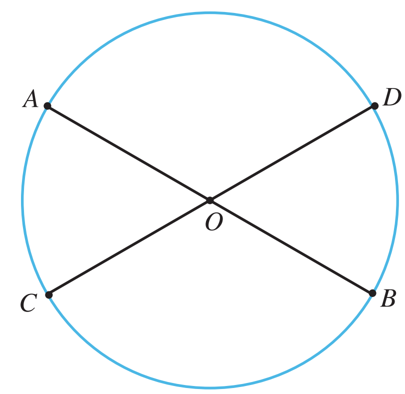
4º) Calculate the length of arc $\overset{\curvearrowright}{AC}$.

5º) Calculate the area of the circular sector $OABC$.

6º) Show that the arcs $\overset{\curvearrowright}{AC}$ and $\overset{\curvearrowright}{BD}$ have the same measure.

2 $[AB]$ and $[CD]$ are two diameters of a circle (C).

Show that $\overset{\frown}{AD} = \overset{\frown}{BC}$. **(2 points)**



3 $C\,(O\,,r)$ and $C'\,(O'\,,r)$ are two secant circles intersecting at A and B.
A line passing through B cuts the circles again at E and F respectively.

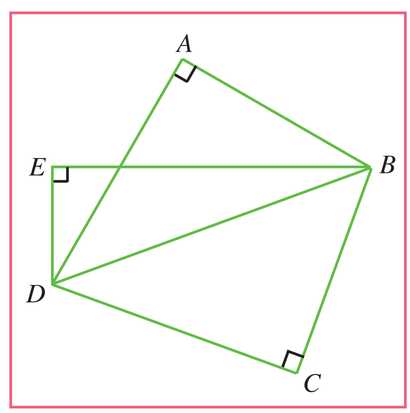
1º) What is the nature of quadrilateral $AO'BO$? Justify. **(2 points)**

2º) Show that triangle AEF is isosceles. **(2 points)**

4 In the following figure, show that the points A, B, C, D and E belong to the same circle (C).

(3 points)

Indicate the center and the radius of (C).



5 (C) is a circle of center O and diameter $[BF]$.
Consider A a point of (C) ($\overset{\frown}{AB} < \overset{\frown}{AF}$).
The tangent to (C) at F cuts (BA) at E.

1º) Show that $\widehat{FEA} = \widehat{AFB}$. **(3 points)**

2º) Suppose that $\overset{\frown}{AB} = 30°$; calculate $\widehat{BEF}$ and $\widehat{AFE}$. **(2 points)**

20 STATISTICS

Objective

To display the data in a frequency polygon.

CHAPTER PLAN

COURSE

1. Reminder of the statistics vocabulary
2. Cumulative frequency
3. Cumulative relative frequency
4. Graph representation and circular diagram
5. Commented exercise

EXERCISES AND PROBLEMS

TEST

COURSE

1 REMINDER OF THE STATISTICS VOCABULARY

Activity

These are the scores over 20 obtained by 36 students in a class :

12	8	12	15	15	8	6	13	14
6	11	10	13	12	17	6	15	13
12	10	8	5	15	14	13	12	11
10	12	14	6	8	10	11	13	12

1°) Complete the following table.

Score	5	6	8	10	11	12	13	14	15	17	
Frequency	1				3	7					Total 36

2°) a) How many students got 6 ? 11 ? 15 ?

b) What does the number 7 represent in this table ?

c) How many students got more than 12 ? Less than 10 ?

d) How many students got 13 and less ? 14 and more ?

e) How many students got at most 8 ? 11 at least ?

f) How many students got the average 10 ?

3°) The relative frequency equals to the score 11 is : $\frac{3}{36} = 0.0833$ or 8.33%.

8.33 is the cumulative percentage of the score 11.

Complete the following table.

Score	5	6	8	10	11	12	13	14	15	17	
Cumulative percentage					8.33						Total 100

a) What is the relative cumulative percentage of the scores below 8 ? below 13 ?

b) What is the percentage of the students who got the average 10 ?

⊙ The group, on which a statistical study is carried, is called **population**.

⊙ Each element of the population is called **individual**.

- The phenomenon studied on a population is called **character**. The character may be **quantitative** (measurable) or **qualitative**.
- The number of individuals in a population is called **total frequency**.
- The number of individuals that verifies the value of a character is the **frequency**.
- The ratio $\dfrac{\textbf{Frequency}}{\textbf{total frequency}}$ is the **relative frequency**.
- The relative frequency is a number between 0 and 1.
- The sum of the relative frequencies is equal to 1.
- The relative frequency may be expressed in percentage if we multiply it by 100.
- The values of the character are placed in increasing order.

2 CUMULATIVE FREQUENCY

Consider a quantitative character.

The cumulative frequency corresponding to a value x of the character, **is the sum of numbers of individuals whose value of the character is less than or equal to x.**

In activity 1, the cumulative frequency of the value 11 is :

$1 + 4 + 4 + 4 + 3 = 16.$

3 CUMULATIVE RELATIVE FREQUENCY

The cumulative relative frequency, corresponding to a value x of the character, is **the sum of the relative frequencies whose value of the character is less than or equal to x.**

This cumulative relative frequency is also the ratio of the cumulative frequency to the total frequency. This relative frequency may be expressed in percentage if we multiply it by 100.

In activity 1, the cumulative relative frequency of the value 11 is :

$\dfrac{16}{36} = 0.4444$; therefore 44.44 %.

4 GRAPH REPRESENTATION AND CIRCULAR DIAGRAM

EXAMPLE

Evaluate the distribution of tourists in Lebanon depending on the following lodging type :

residency with parents or friends 45 % ,

renting 25 % ,

hotel 30 % .

Represent this distribution by a **circular diagram**.

⊙ Divide the disc into sectors whose angles are proportional to the percentages.

360° corresponds to 100%.

Therefore, the proportionality coefficient is :

$$K = \frac{360}{100} = 3.6 .$$

⊙ You may also divide the disc into sectors whose angles are proportional to the frequency.

360° corresponds to the total frequency.

Therefore the proportionality coefficient is :

$$K = \frac{360}{\text{total frequency}} .$$

⊙ Every sector is defined by an angle whose vertex is the center of the disc.

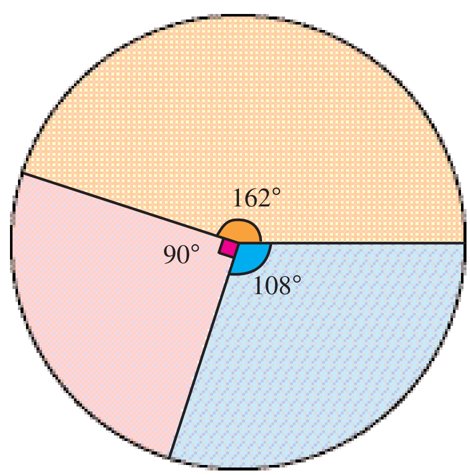
⊙ Draw the angles using a protractor.

The distribution of the tourists is summarized in the following table :

lodging type	Parents or friends	Renting	Hotel	Total
Percentage	45	25	30	100
Angles in degrees	162	90	108	360

K=3.6

The proportionality coefficient is $K = \frac{360}{100} = 3.6$.



The angle corresponding to :

⊙ 45 % is $45 \times 3.6 = 162°$,

⊙ 25 % is $25 \times 3.6 = 90°$,

⊙ 30 % is $30 \times 3.6 = 108°$.

5 COMMENTED EXERCISE

Statement

A survey, done with the students of two classes about the number of books they read during the summer vacation, has shown the following results :

25 students read 2 books,	**15 students read 3 books,**
8 students read 4 books,	**2 students read 5 books.**

1º) What is the studied population ? What is the individual ?

2º) What is the studied character ? Is it qualitative or quantitative ?

3º) What is the total frequency of students ?

4º) a) Draw the table of frequency and of cumulative frequency.

b) What is the number of students who read at most 3 books ? At least 4 books ?

5º) a) Draw the table of relative frequencies, frequencies in percentage, and cumulative relative frequencies in percentage.

b) What is the percentage of students who read at least 3 books ? 4 books and more ?

6º) Represent this survey by a circular diagram.

Solution

1º) The studied population is the group of students of two classes. The individual is each student in these two classes.

2º) The studied character is the number of books read in summer. This character is quantitative.

3º) The total frequency of students is : $25 + 15 + 8 + 2 = 50$ students.

4°) a)

Number of books read	2	3	4	5	Total
Frequency	25	15	8	2	50
Cumulative frequency	25	40	48	50	

$40 = 25 + 15$

b) The number of students who read at most 3 books is : 40.

The number of students who read 4 books and more is : $8 + 2 = 10$.

5°) a) To calculate the relative frequency of a value, we calculate the ratio of the frequency of this value with the total frequency.

For instance, the relative frequency of the value 3 is :

$\frac{15}{50} = 0.3$ or $0.3 \times 100 = 30$ %.

Number of books read	2	3	4	5	Total
Relative frequency	0.5	0.3	0.16	0.04	1
Relative frequency in %	50	30	16	4	100
Cumulative relative frequency in %	50	80	96	100	

$50 + 30 = 80$

b) According to the previous table, 80% of the students read at most 3 books .

This percentage can be found by multiplying by 100 the ratio of the cumulative relative frequency to the value 3 divided by the total frequency :

$$\frac{40}{50} \times 100 = 80 \text{ \%}.$$

The percentage of students who read 4 books and more is :

$16 + 4 = 20$; therefore 20 %.

6°) To represent this survey by a circular diagram, we divide the disc into four sectors.

Each sector is defined by an angle whose vertex is the center of the disc and which is proportional to the corresponding frequency. The proportionality coefficient is :

$$K = \frac{360}{\text{total frequency}} = \frac{360}{50} = 7.2 \,.$$

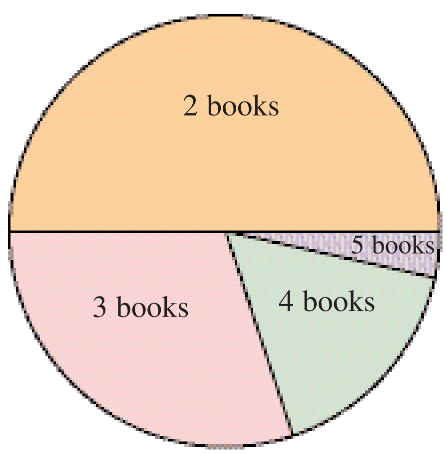
For instance, the angle corresponding to 3 books is :

$15 \times 7.2 = 108°$.

The following table shows the angles of different sectors.

Number of books read	2	3	4	5	Total
Frequency	25	15	8	2	50
Angle in degrees	180	108	57.6	14.4	360

K=7.2



EXERCISES AND PROBLEMS

Test your knowledge

1 Answer by true or false.

The following table shows the number of children of the families in a building.

Number of children	0	1	2	3	4
Frequency (Number of families)	2	5	8	4	1
Cumulative frequency	2	7	15	19	20

1º) The studied population is the group of families in a building.

2º) The studied character is qualitative.

3º) The total frequency is 20.

4º) 15 families have at least 2 children.

5º) 19 is the cumulative frequency of the value 4.

6º) The relative frequency to the value 2 is 0.4 or 40%.

7º) 95% of the families have at least 3 children.

8º) In a circular diagram :
a) the disc is divided into 5 sectors.
b) The proportionality coefficient is $K = \dfrac{360}{20} = 18$.
c) The angle of the sector corresponding to the value 1 is 80°.

2 The world production of petroleum is divided in the following manner : 25% for the United States (U.S.A), 30% for the Middle East (M.E) and 45% for the other countries.

1º) Complete the following table.

Population zone	U.S.A	M.E	Others	Total
Percentage	25			100
Angle in degrees	90			360

$K=$...........

2º) Represent this world production by a circular diagram.

3 The studied population is the group of 30 students in a class. The studied character is the number of brothers and sisters.

1º) Complete the following table; then draw a circular diagram.

Value	0	1	2	3	4 and more	Total
Frequency	8	12	5	3	2	
Frequency (%)						100
Angle in degrees						360

2º) What is the number of students who have 2 brothers or sisters at most ? What percentage of the class do they represent ?

4 A survey, done with the students of all classes in a cycle in a school about the color of their eyes, showed the following results :

25 students have blue eyes,
20 students have green eyes,
75 students have black eyes,
100 students have brown eyes.

1º) What is the studied population ?

2º) What is the studied character ? Is it qualitative or quantitative ?

3º) What is the total frequency of students ?

4º) Complete the following table; then draw a circular diagram to show this distribution.

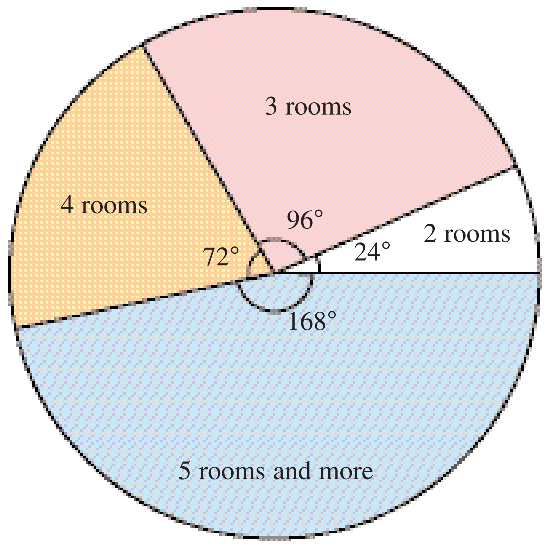
Eyes color	Blue	Green	Black	Brown	Total
Frequency	25	20	75	100	220
Angle in degrees					360

5º) Reproduce the preceding table by replacing the angles with the relative frequencies.

5 The adjacent circular diagram shows the results of a statistical survey done with 60 houses in a neighborhood.

The studied character is the number of main rooms.

Draw the table showing the frequency, the cumulative frequency, the relative frequencies, and the cumulative relative frequencies of the values of the studied character.



6 The results of the delegates election in a class, where all the students have voted, are summarized in the following table :

Candidate name	Ziad	Walid	Karim
Number of votes	15	6	3

1°) Draw the table that shows the percentage of the votes obtained by each candidate.

2°) Represent the distribution of the votes in a circular diagram.

7 All the students in one cycle in a school chose one activity :

110 sports, 80 drawing, and 50 music.

Represent the distribution of the students in this cycle by a circular diagram depending on the chosen activity.

8 **«I am a good boy to some extent».**

1°) Count the total number of letters in this sentence.

2°) Draw a table showing the number of each letter and its percentage.

3°) What is the percentage of vowels ?

4°) **a)** Which vowel has the highest frequency ?
b) Which consonant has the highest frequency ?

For seeking

9 The scores of the students of a class on a math test are distributed in the following way :

5% got 18 , 20% got 15 , 25% got 12 , 37.5% got 10 and 12.5% got 7.

1°) Draw a table of the cumulative relative frequencies in percentage.

2°) What is the percentage of students who got 10 and more ?

3°) Draw a circular diagram to show this distribution.

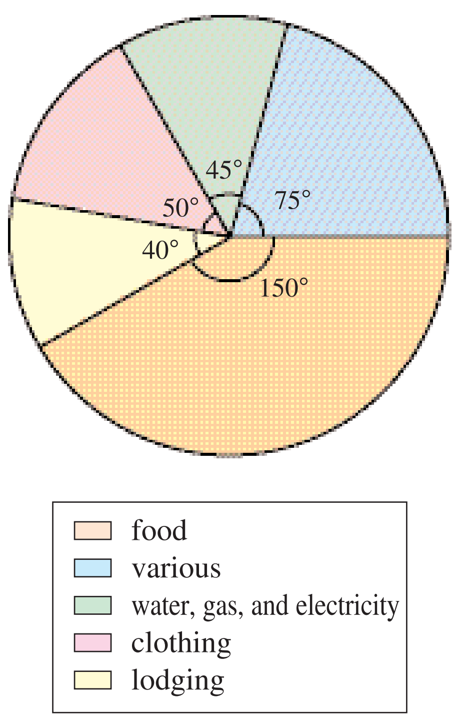
10 The adjacent diagram shows the distribution of the expenses of a family.

1°) In a table, represent the percentage of this distribution.

2°) Knowing that the income is 750 000 L.L. per month, calculate the sum of money dedicated to each part in this distribution.

3°) Another man earns 900 000 L.L. per month.
He spends 400 000 L.L. for food,
180 000 L.L. for lodging,
100 000 L.L. for clothing,
120 000 L.L. for «gas - electricity - water», and the rest for various expenses.

Represent these in a circular diagram.



11 In a bookshop, 20% of the books are religious, 30% are literary, 45% scientific, and 5% have various topics.

1º) What is the studied population ? What is the character ? Is it qualitative or quantitative ?

2º) Draw a circular diagram to show this distribution.

12 A fruit cocktail is made of :

1 volume of sugar cane syrup,

3 volumes of lemon juice,

6 volumes of orange juice.

1º) What is the quantity in cl in each of the components in 100 cl of the beverage ?

2º) Complete the table below, which shows the quantity in cl of each component in one liter (100 cl) of this beverage.

Component	Syrup	Lemon	Orange	Total
Number of volumes	1	3	6	10
Quantity in cl				100

3º) Draw a circular diagram to show this composition.

TEST

1 A dice, whose sides are numbered 1, 2, 3, 4, 5 and 6, is thrown.

The following table shows the number of appearances of each side.

Number of the side	1	2	3	4	5	6
Frequency	3	6	8	2	7	4

Answer by true or false.

1º) The value 5 has the most frequent appearance.

2º) The relative frequency of the value 2 is $\frac{1}{5}$ or 20%.

3º) 4 is the relative frequency of the value 6.

4º) The relative frequency of appearance of an even number is : $\frac{6+2+4}{30} = \frac{2}{5}$ or 40%.

5º) The relative frequency of appearance of an odd number is 50%.

6º) In a circular diagram, the angle corresponding to the value 2 is 20°.

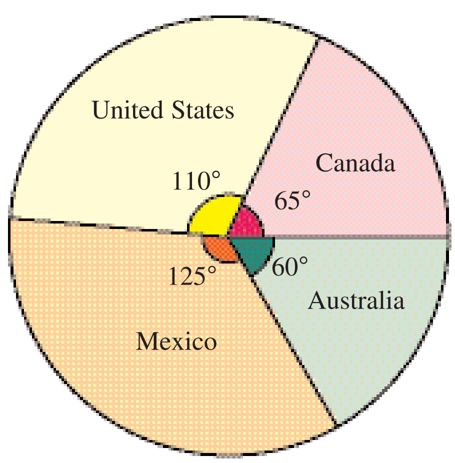
(6 points)

2 The production of money in 1993 is of 7200 tons in the following four countries : Canada, the United States, Mexico and Australia.

The distribution of this production is shown in the adjacent circular diagram.

Draw a table, showing the production, in tons, of each country and the percentage of this production.

(6 points)



3 3000 students of a school are distributed in the following way : 1020 students in the preschool, 30% in the elementary, 22% in the intermediate, and the rest in the secondary.

1º) Find the number of students in the secondary. **(2 points)**

2º) Complete the following table :

Level	Preschool	Elementary	Intermediate	Secondary
Frequency	1020			
Cumulative frequency				
Relative frequency (in %)		30	22	
Cumulative relative frequency (in %)				

(4 points)

3º) Represent this distribution in a circular diagram. **(2 points)**

21 COORDINATES OF THE MIDPOINT OF A SEGMENT

Objective

To calculate the coordinates of the midpoint of a segment in a plane.

CHAPTER PLAN

COURSE

1. Average of many numbers
2. Abscissa of the midpoint of a segment
3. Coordinates of the midpoint of a segment in an orthonormal system of a plane

EXERCISES AND PROBLEMS

TEST

COURSE

1 AVERAGE OF MANY NUMBERS

Activity

These are the mathematics grades that Ziad and Walid got during the first term :

Assignment nº	1	2	3	4	5
Scores of Ziad	13	10	14	12	6
Scores of Walid	12	9	13	11	14

In this activity, the objective is to know which one of them is better in mathematics.

Therefore, we proceed in this way :

For Ziad : $\frac{13 + 10 + 14 + 12 + 6}{5} = 11$.

The denominator 5 represents the number of assignments.

11 is called the average of Ziad's grades.

Do the same to find the average of Walid's grades.

Deduce who is better in mathematics.

Rule

The average of *n* numbers is the quotient of their sum by *n* .

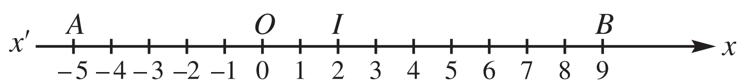
Application 1

Find the average of the following numbers.

1º) 3.7 and 5.3. **2º)** – 3 ; 5 and 7. **3º)** 2 ; 3 ; 4 and 11.

2 ABSCISSA OF THE MIDPOINT OF A SEGMENT

A and B are two points on an axis $x'Ox$ of respective abscissas $\overline{OA} = x_A$ and $\overline{OB} = x_B$.



The abscissa of the midpoint I of segment $[AB]$ is the average of the numbers x_A and x_B .

If I is the midpoint of $[AB]$, then $\overline{OI} = \dfrac{\overline{OA} + \overline{OB}}{2}$ or $x_I = \dfrac{x_A + x_B}{2}$.

EXAMPLE

A and B are two points of an axis $x'Ox$ such that $\overline{OA} = -5$ and $\overline{OB} = +9$.

The abscissa of I, the midpoint of $[AB]$ is : $\overline{OI} = \dfrac{-5+9}{2} = 2$.

Application 2

1°) On the axis $x'Ox$, place the points A, B and C of respective abscissas -3 ; $+5$ and -1.

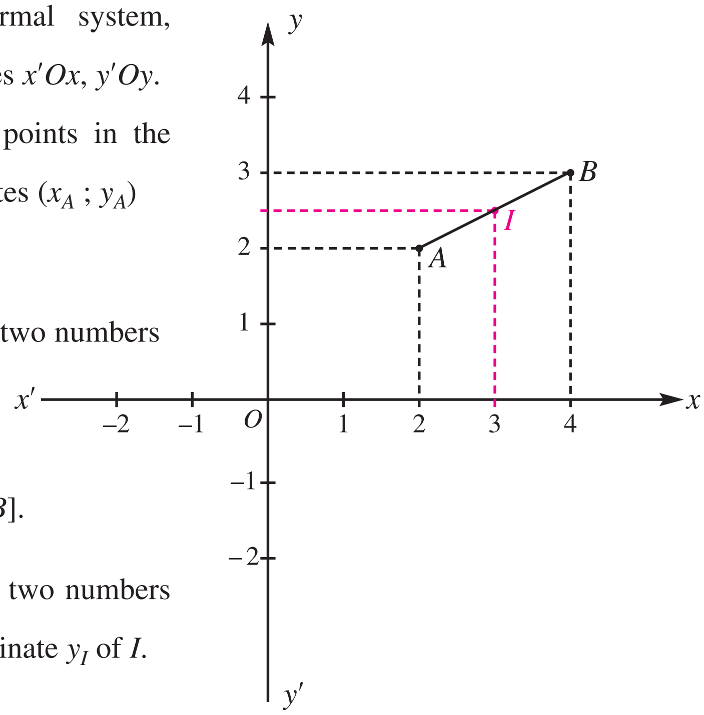
2°) Calculate the abscissas of these points : I, the midpoint of $[AB]$; J, the midpoint of $[AC]$; K the midpoint of $[BC]$ and L, the midpoint of $[IJ]$.

3 COORDINATES OF THE MIDPOINT OF A SEGMENT

Draw an orthonormal system, in a plane, with axes $x'Ox$, $y'Oy$.
A and B are two points in the system of coordinates $(x_A\ ;\ y_A)$ and $(x_B\ ;\ y_B)$.

The average of the two numbers x_A and x_B is the abscissa x_I of I, the midpoint of $[AB]$.

The average of the two numbers y_A and y_B is the ordinate y_I of I.



If I is the midpoint of $[AB]$, then : $x_I = \dfrac{x_A + x_B}{2}$ and $y_I = \dfrac{y_A + y_B}{2}$.

EXAMPLE

A and B are two points of the system, such that : $A\ (2\ ;\ 2)$ and $B\ (4\ ;\ 3)$.

If I is the midpoint of $[AB]$, then : $x_I = \dfrac{2+4}{2} = 3$ and $y_I = \dfrac{2+3}{2} = \dfrac{5}{2}$,

hence $I\ (3\ ;\ \dfrac{5}{2})$.

Application 3

1°) In an orthonormal system of axes $x'Ox$, $y'Oy$, place the points $A\ (-2\ ;\ 3)$, $B\ (3\ ;\ -1)$ and $C\ (-1\ ;\ 4)$.

2°) Calculate the coordinates of these points : I midpoint of $[AB]$, J midpoint of $[BC]$ and K midpoint of $[IJ]$.

EXERCISES AND PROBLEMS

Test your knowledge

1 Answer by true or false.

1º) The average of 7 and – 5 is 1.

2º) 3 is the average of 2 ; 5 and 3.

3º) 0 is the average of 3 and – 3.

4º) The average of many numbers is a positive number.

5º) *A* and *B* are two points on an axis of respective abscissas – 3 and 5.
The point *I* with abscissa 2 is the midpoint of [*AB*].

6º) *A* and *B* are two points on an axis with respective abscissas – 1 and 1.
The origin *O* of the axis is the midpoint of [*AB*].

7º) Consider *A* (1 ; 0) and *B* (3 ; 2) two points in an orthonormal system of origin *O*.
The point *I* (2 ; 2) is the midpoint of [*AB*].

2 Find the average of the following :

1º) – 6 and 4.8.

2º) – 4 ; – 3 and – 2.

3º) – 3 ; – 2.1 and 4.

4º) – 2.1 ; 0 ; 2.5 and 6.

3 The scores obtained by Sami and Nabil on six French assignments are summarized in the following table.

Assignment nº	1	2	3	4	5	6
Scores of Sami	10	11	7	9	13	14
Scores of Nabil	11	9	5	8	14	12

1º) What is the average of Sami's scores ? Of Nabil's scores ?

2º) Chadi was absent for the third assignment, and he got the following scores : 10 , 8 , 7 , 13 and 11 on the others.

Find the average of Chadi and deduce who is the best among the three students.

4 A and B are two points on an axis $x'Ox$. Calculate the abscissa x of I, the midpoint of $[AB]$, in each of the following cases :

1º) $A\,(4)$ and $B\,(-2)$.

2º) $A\,(1.5)$ and $B\,(-5.3)$.

3º) $A\,(-5.1)$ and $B\,(-3.5)$.

4º) $A\,(2.1)$ and $B\,(-6.3)$.

5 Draw an orthonormal system of axes $x'Ox$, $y'Oy$. Calculate the coordinates of point I, the midpoint of $[AB]$, in each of the following cases :

1º) $A\,(2\,;\,5)$ and $B\,(3\,;\,6)$.

2º) $A\,(-1.5\,;\,0)$ and $B\,(0\,;\,2.5)$.

3º) $A\,(-2.1\,;\,3)$ and $B\,(2.1\,;\,-3)$.

4º) $A\,(-3\,;\,-4)$ and $B\,(-1.5\,;\,-2.5)$.

6 Consider an orthonormal system of axes $x'Ox$, $y'Oy$.

1º) Place the points A (2 ; 1) , B (– 1 ; 3) , C (– 7 ; 5) and D (4 ; 3).

2º) Find the coordinates of I, the midpoint of [AB] and of J, the midpoint of [CD]. Plot the points I and J.

For seeking

7 A and B are two points on an axis $x'Ox$. The point I of abscissa – 2 is the midpoint of segment [AB].

1º) Calculate the abscissa of point A, if the abscissa of B is – 5.

2º) Calculate the abscissa of point B, if the abscissa of A is 3.

8 Consider an orthonormal system of axes $x'Ox$, $y'Oy$. N is the symmetric of M with respect to I. Calculate the coordinates of N in each of the following cases :

1º) M (1 ; 5) and I (4 ; 3).

2º) M (– 3 ; – 2.4) and I (– 2 ; 2.2).

9 Consider an orthonormal system of axes $x'Ox$, $y'Oy$.

1º) Place the points A (2 ; 3) and B (– 1 ; 4).

2º) Calculate the coordinates of C the symmetric of A with respect to O. Plot point C .

3º) Plot the point D (1 ; – 4) and calculate the coordinates of the midpoint of [BD].

4º) Deduce the nature of quadrilateral $ABCD$.

10 **1º)** Find the coordinates of I the center of the circle of diameter [AB] with A (– 2 ; 4) and B (6 ; 6), in an orthonormal system.

2º) Then calculate the coordinates of J the center of the circle of diameter [AI].

3º) Construct the two circles.

TEST

1 **1º)** On an axis $x'Ox$, Place the points A, B, C and D of respective abscissas 2 ; 3 ; – 4 and – 1.

2º) Calculate the abscissa of I the midpoint of $[AB]$, the abscissa of J the midpoint of $[CD]$, and the abscissa of K the midpoint of $[IJ]$. What do you notice ? **(4 points)**

2 On an axis $x'Ox$, place the points A, B, C and D of respective abscissas – 1.4 ; – 3.2 ; 2.1 and 3.9.

Show that the segments $[AC]$ and $[BD]$ have the same midpoint. **(3 points)**

3 In an orthonormal system, consider the points A (2 ; 5), B (– 2 ; 2), C (3 ; 1) and D (7 ; 4).

1º) Calculate the coordinates of point I, the midpoint of $[AC]$ and of point J, the midpoint of $[BD]$. What do you notice ? **(3 points)**

2º) What is the nature of quadrilateral $ABCD$? **(2 points)**

4 The average of the ages of Fadi, Nabil and Leyla is 15 years. Fadi is 12 years old and Nabil is 17.

How old is Leyla ? **(4 points)**

5 **1º)** In an orthonormal system, construct the circle of center I (1 ; 3) and radius 2 units. **(1 point)**

2º) Are the points A (1 ; 5) and C (3 ; 3) on this circle ? **(1 point)**

3º) Calculate the coordinates of B diametrically opposite to A and the coordinates of D, diametrically opposite to C. **(2 points)**

22 VECTOR AND TRANSLATION

Objectives

1. To represent geometrically a vector.

2. To identify the vector of a translation.

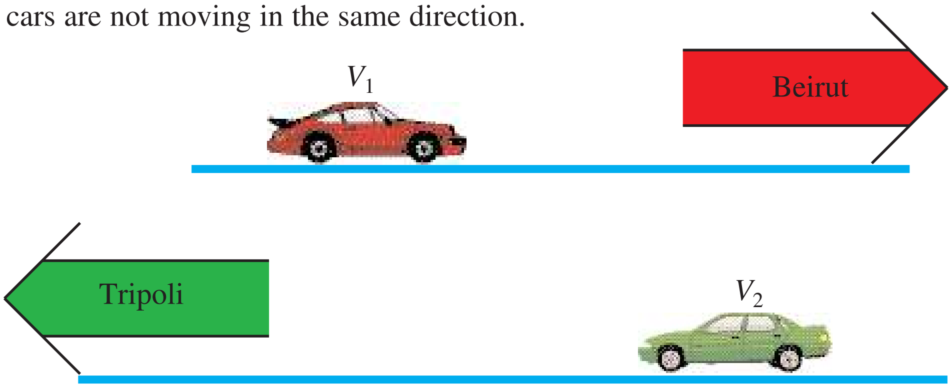
3. To draw the translation of a given figure.

CHAPTER PLAN

1 DIRECTION AND SENSE

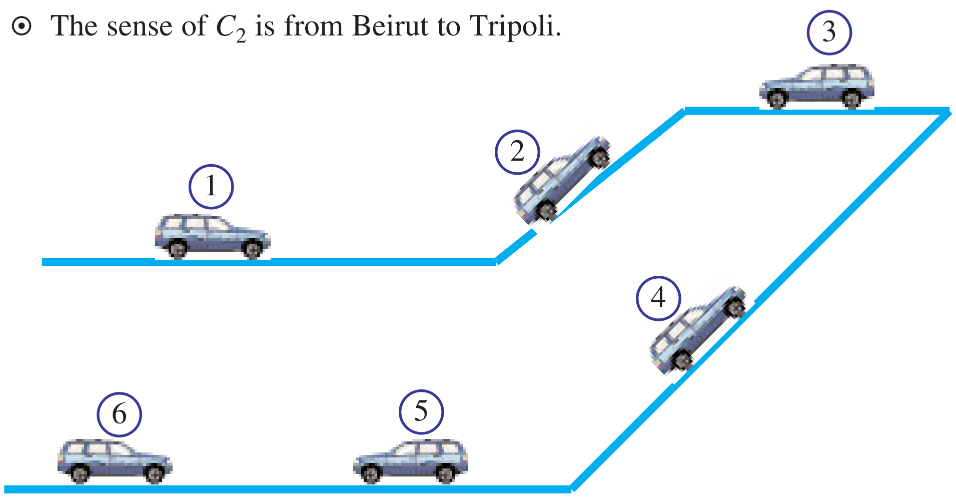
Activity

By looking at the diagram below and in the usual language, we say that the cars are not moving in the same direction.



But, in mathematics, it's different : both cars C_1 and C_2 are moving in the **same direction** but in **opposite sense.**

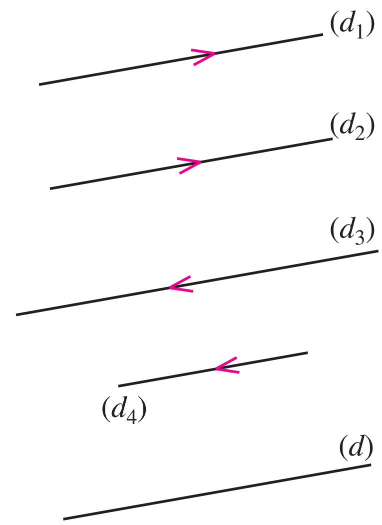
- The sense of C_1 is from Tripoli to Beirut.
- The sense of C_2 is from Beirut to Tripoli.



Using their numbers, indicate the cars that are moving :

- in the same direction.
- in the same direction and in the same sense.
- in the same direction but in opposite sense.

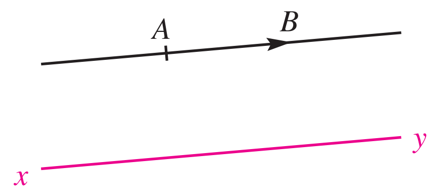
Definitions



⊙ When two lines are **parallel**, they have the **same direction**.
(d_1), (d_2), (d_3) and (d_4) have the same direction, which is that of (d).

⊙ When a **direction is given**, a **sense** can be chosen. Then there are two possibilities as shown in the diagram.
A sense was chosen on (d_1) and (d_2), and another sense, opposite to the first, was chosen on (d_3) and (d_4).

2 VECTOR



On a segment $[AB]$, the following is defined :

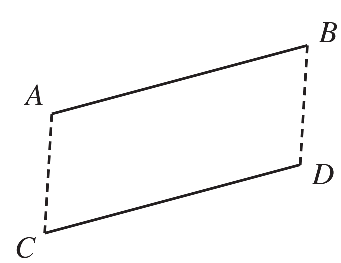
⊙ **a direction** : the direction of the line (AB) or of any line (xy) parallel to (AB).

⊙ **a sense** : the sense moving from A to B.

⊙ **a length (magnitude)** : the length of the segment $[AB]$.
These three characteristics define **vector** $\boldsymbol{AB}$, denoted by $\overrightarrow{\boldsymbol{AB}}$. $\boldsymbol{A}$ is its **origin** and $\boldsymbol{B}$ its **extremity.**

Attention !
The arrow above the vector is always represented from left to right $\longrightarrow$.

3 EQUAL VECTORS



Consider the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

The equality $\overrightarrow{\boldsymbol{AB}} = \overrightarrow{\boldsymbol{CD}}$ means :

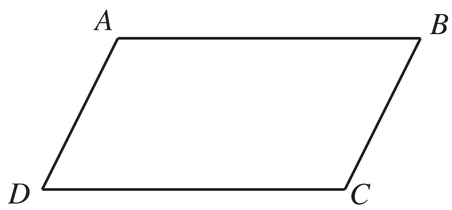
⊙ **(AB) is parallel to (CD)**, ($\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction).

⊙ **$\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same sense**.

⊙ $\boldsymbol{AB = CD}$ ($\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same length).

Consequently the equality $\overrightarrow{\boldsymbol{AB}} = \overrightarrow{\boldsymbol{CD}}$ is equivalent to ***ABDC*, which is a parallelogram.**

Application 1



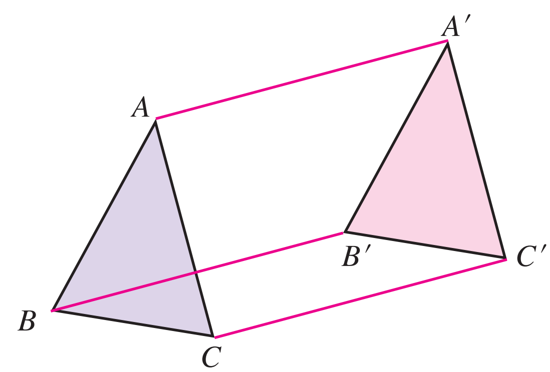
$ABCD$ is a parallelogram.

Complete :

- $\overrightarrow{AB} = \ldots$; $\overrightarrow{AD} = \ldots$; $\ldots = \overrightarrow{BA}$; $\overrightarrow{CB} = \ldots$
- $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same .... and .... sense .
- $\overrightarrow{AD}$ and .... have the same direction.
- $\overrightarrow{CB}$ and .... have opposite sense.
- .... and $\overrightarrow{CD}$ have the same length.

4 VECTOR AND TRANSLATION

The triangle $A'B'C'$ is obtained by sliding triangle ABC :



- in the direction of the line (AA'),
- in the sense from A to A',
- of a length equal to the length of segment $[AA']$.

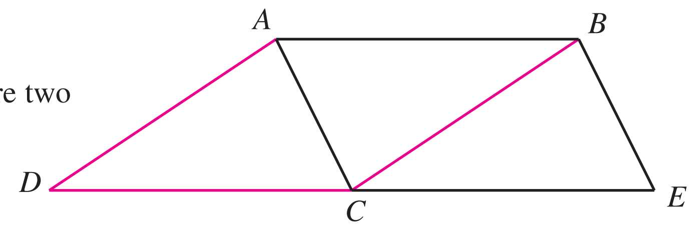
We say that : **$A'B'C'$ is the image of ABC by the translation of vector $\overrightarrow{AA'}$** .

We may also say that this translation is of vector $\overrightarrow{BB'}$ or $\overrightarrow{CC'}$, because $\overrightarrow{BB'} = \overrightarrow{AA'}$ and $\overrightarrow{CC'} = \overrightarrow{AA'}$.

The points **A', B' and C'** are called the **images** of the points A, B and C, **by the translation of vector $\overrightarrow{AA'}$** or they are the **translates** of A, B and C.

Application 2

ABCD and *ABEC* are two parallelograms.



1º) What are the respective images of C and D by the translation of vector $\overrightarrow{AB}$?

2º) Find a translation vector that transforms D into A and C into B.
Is there another vector ?

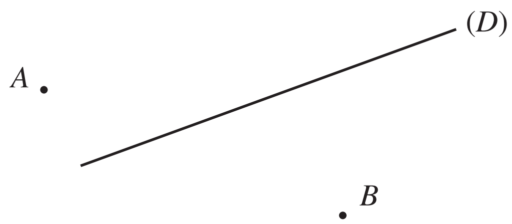
3º) What is the image of triangle ADC by the translation of vector $\overrightarrow{AB}$?

EXERCISES AND PROBLEMS

Test your knowledge

1 **1º)** From the points A and B, draw the lines (D_1) and (D_2) having the same direction as (D).

2º) How many senses can be defined on each of the lines ?



2 Consider the figures 1, 2, 3 and 4.

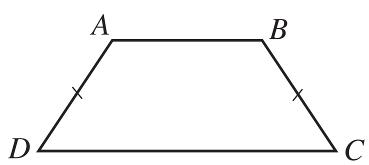


Fig. 1

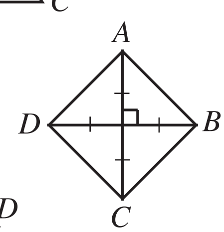


Fig. 2

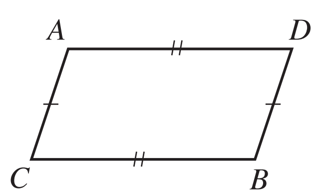


Fig. 3

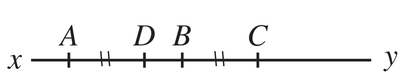
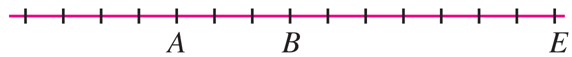


Fig. 4

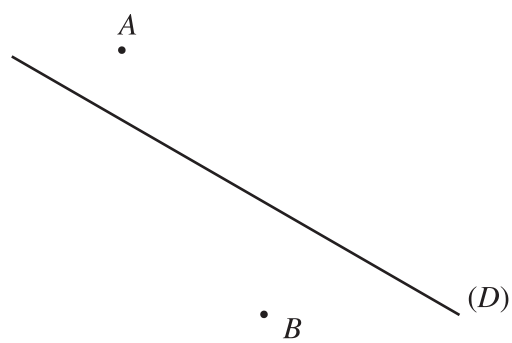
Which figures have :
$\overrightarrow{AD} = \overrightarrow{BC}$.

3 On the adjacent figure place the points C, D and F such that :
$\overrightarrow{AC} = \overrightarrow{BE}$, $\overrightarrow{CD} = \overrightarrow{BA}$, $\overrightarrow{FB} = \overrightarrow{BE}$.



4 **1º)** From the points A and B, draw the vectors $\overrightarrow{AC}$ and $\overrightarrow{BE}$ having the same direction as (D), of a length 4 cm and having opposite senses.

2º) What is the nature of quadrilateral $ACBE$?



5 **1º)** A, B and C are three fixed and non-collinear points.

a) What is the image of A by the translation of vector $\overrightarrow{AB}$?

b) What is the image of B by the translation of vector $\overrightarrow{BC}$?

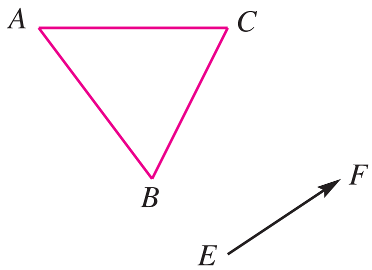
2º) The image of a point E, by a translation, is the point F.

What is the vector of this translation ?

3º) I is the image of M, by the translation of vector $\overrightarrow{LE}$.

What is the nature of quadrilateral $MIEL$?

6 Construct the image of triangle ABC by the translation of vector $\overrightarrow{EF}$.



7 Answer by true or false.

1º) To designate the vector $B \longleftarrow A$, we write $\overrightarrow{AB}$.

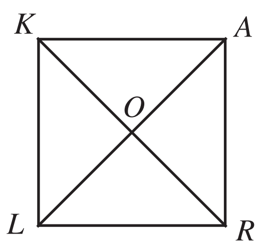
2º) **a)** If $\overrightarrow{AB} = \overrightarrow{CD}$, then $AB = CD$.

b) If $\overrightarrow{AB} = \overrightarrow{CD}$, then (AB) is parallel to (CD).

c) If $\overrightarrow{AB} = \overrightarrow{BC}$, then B is the midpoint of $[AC]$.

d) If $\overrightarrow{AB} = \overrightarrow{DC}$, then $\overrightarrow{AC} = \overrightarrow{BD}$.

3º) $KARL$ is a square of center O.



a) (KA) and (RL) have the same direction.

b) The vectors $\overrightarrow{KA}$ and $\overrightarrow{AR}$ have the same direction.

c) $\overrightarrow{OL} = \overrightarrow{AO}$.

d) A is the image of K by the translation of vector $\overrightarrow{LR}$.

e) The translation of vector $\overrightarrow{KL}$ is the only one that transforms A into R.

f) The image of L by the translation of vector $\overrightarrow{OA}$ is R.

g) The image of K by the translation of vector $\overrightarrow{OR}$ is O.

For seeking

8 $ABCD$ is a square of side 3 cm. Construct the image of this square by each of these translations :

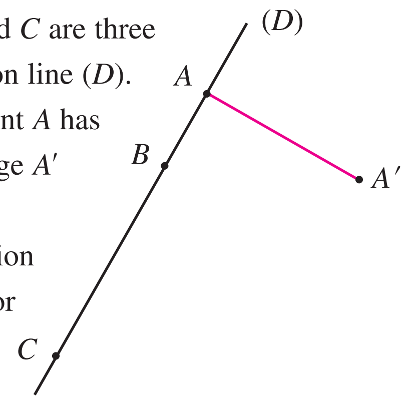
a) of vector $\overrightarrow{AB}$,

b) of vector $\overrightarrow{BC}$,

c) of vector $\overrightarrow{AC}$.

In each case, find the nature of the obtained quadrilateral.

9 A, B and C are three points on line (D). The point A has for image A' by the translation of vector $\overrightarrow{AA'}$.



1º) Construct the images B' and C' of points B and C, by this translation.

2º) What is the nature of the quadrilaterals $AA'B'B$ and $AA'C'C$? Show that the points A' , B' and C' are on the same line (D').

3º) What is the image of line (D) by this translation ?

10 Consider a triangle ABC and the translation by vector $\overrightarrow{AC}$.

1º) What is the image of A by this translation ?

2º) Construct the images B' and C' of B and C , respectively, by this translation.

3º) Show that $BB' = CC'$.

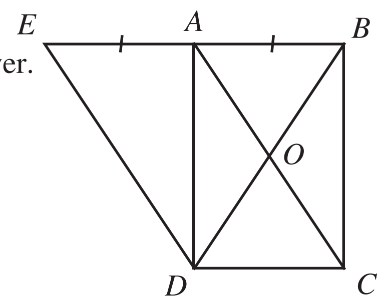
4º) What is the nature of the quadrilaterals $ACB'B$ and $BCC'B'$?

5º) Deduce that triangles ABC and $CB'C'$ are congruent.

TEST

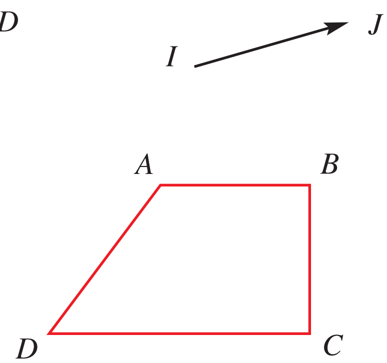
1 Consider the adjacent figure.
Complete the following table by checking the correct answer.

	True	False
$\overrightarrow{AB} = \overrightarrow{CD}$		
$\overrightarrow{AC} = \overrightarrow{ED}$		
$\overrightarrow{AC} = \overrightarrow{BD}$		
$AC = BD$		
$\overrightarrow{OA} = \overrightarrow{DE}$		
$\overrightarrow{OA} = \overrightarrow{OC}$		



(3 points)

2 Construct the image of the trapezoid $ABCD$ by the translation of vector $\overrightarrow{IJ}$.



(3 points)

3 $ABCD$ is a rhombus.

1º) Construct point E the image of C by the translation of vector $\overrightarrow{AD}$.

2º) Justify the equality of vectors $\overrightarrow{AD}$, $\overrightarrow{BC}$ and $\overrightarrow{CE}$.

3º) What can you say about point C with respect to segment $[BE]$? **(6 points)**

4 Consider a circle (C) of center O and radius 4 cm. Place a point O' at 6 cm from O.

1º) What is the image of O by the translation of vector $\overrightarrow{OO'}$?

2º) Let A be a point of (C). Construct the image A' of A by this translation. What is the nature of quadrilateral $OO'A'A$?
Complete : $\overrightarrow{OA} = ...$ and $OA = O'A' = ...$ cm.

3º) Draw a circle (C') of center O' and radius 4 cm.
a) Let B be a point of (C) and B' its image by the translation of vector $\overrightarrow{OO'}$. What is the nature of quadrilateral $OO'B'B$?
b) Complete : $\overrightarrow{OB} = ...$ and $OB = O'B' = ...$ cm.
Where is point B' located ? **(8 points)**

23 GEOMETRIC LOCI AND CONSTRUCTIONS

Objectives

1. To find the locus of a point verified by a given property.

2. To use the locus in construction.

CHAPTER PLAN

COURSE

1. Geometric locus

2. Some geometric loci
1º) Perpendicular bisector of a segment
2º) Circle
3º) Circle with fixed diameter
4º) Bisector of an angle
5º) Line forming a constant angle with another fixed line
6º) Line, drawn from a fixed point, parallel to another fixed line

3. Constructions

EXERCISES AND PROBLEMS

TEST

1 GEOMETRIC LOCUS

Definition

A locus is a **line** (straight or curved) formed by all the points that have the **same property**.

The statement of a geometric locus is made of :

- the name of the line forming the locus;
- the property that the points of this line have.

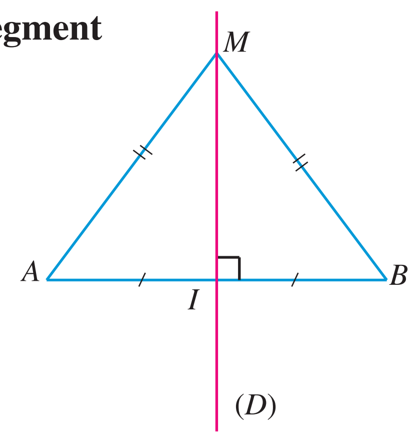
2 SOME GEOMETRIC LOCI

1°) Perpendicular bisector of a segment

A and *B* are two fixed points.

M is a variable point equidistant from *A* and *B*.

The perpendicular bisector (*D*) of segment [*AB*] is the locus of all points *M* equidistant from *A* and *B* .



(*D*) is the perpendicular bisector of [*AB*]

Application 1

E and *F* are two fixed points. (*C*) is a variable circle of center *I* passing through *A* and *B*.

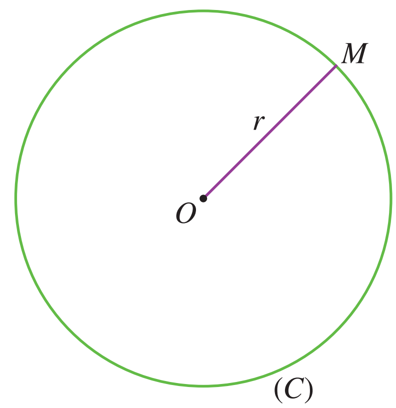
What is the locus of *I* ?

2°) Circle

O is a fixed point.

M is a variable point such that
$OM = r =$ constant.

The circle (C) of center O and radius r is the locus of all points having a constant distance r from O.



(C) is a circle of center O and radius r

Application 2

ABC is a triangle right at A such that A is fixed, B and C variables with $BC = 6$ cm.

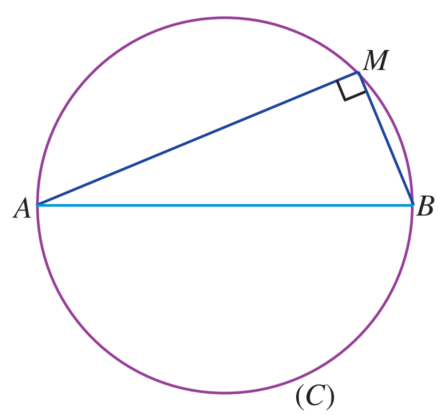
What is the locus of midpoint I of $[BC]$?

3°) Circle with fixed diameter

A and B are two fixed points.

M is a variable point such that
$\widehat{AMB} = 90°$.

The circle (C) of fixed diameter $[AB]$ is the locus of all points M such that $\widehat{AMB} = 90°$.



(C) is a circle of diameter $[AB]$

Application 3

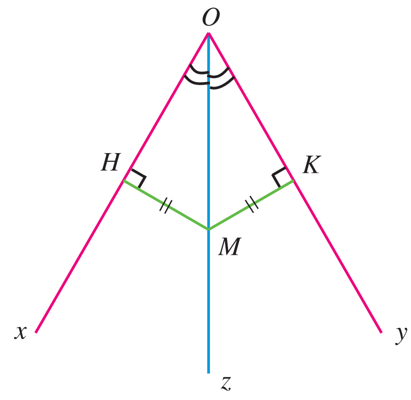
ABC is a triangle such that A and B are fixed and C is variable.

If $[AH]$ is the altitude segment in this triangle, find and construct the locus of H ?

4°) Bisector of an angle

$\widehat{xOy}$ is a fixed angle.

M is a variable point equidistant from $[Ox)$ and $[Oy)$.



$[Oz)$ is the bisector of $\widehat{xOy}$

The bisector $[Oz)$ of the angle $\widehat{xOy}$ is the locus of all points M equidistant from the sides $[Ox)$ and $[Oy)$ of this angle.

Application 4

$\widehat{xAy}$ is a fixed angle. B and C are two variables points respectively on $[Ax)$ and $[Ay)$ such that $AB = AC$. The perpendiculars, drawn from B to $[Ax)$ and from C to $[Ay)$ intersect at I .

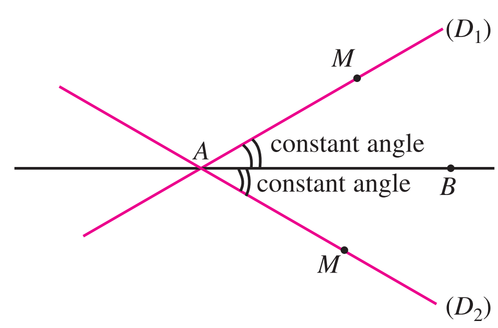
1°) Show that the two triangles ABI and ACI are congruent.

2°) Find the locus of I ?

5°) Line forming a constant angle with another fixed line

(AB) is a fixed line.

M is a variable point such that angle $\widehat{MAB}$ is constant.



(D_1) and (D_2) form the same angle constant with (AB)

The lines (D_1) and (D_2) passing through A, and that form the same angle constant with (AB) constitute the locus of the points M such that $\widehat{MAB}$ is this angle.

Application 5

O is a fixed point and A is a variable point on a fixed line (xy).

Through point A draw the perpendicular to (xy). On (xy) plot $AM = OA$.

1°) Why is point M variable ?

2°) What is the nature of triangle AOM ?

3°) Find and construct the locus of M.

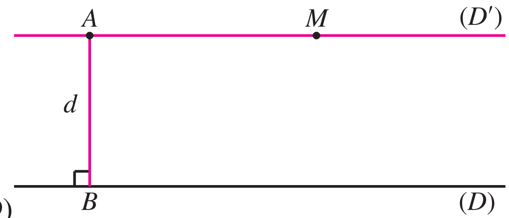
6º) Line, drawn from a fixed point, parallel to another fixed line

(D) is a fixed line.

A is a fixed point.

Therefore, the distance d from A to (D) is constant.

M is a variable point such that the distance from M to (D) is equal to d.



(D) is a fixed line and A is a fixed point. The distance d from A to (D) is constant.

The line (D') passing through A and parallel to (D) is the locus of the points M such that the distance from M to (D) is constant and equal to that from A to (D)

Application 6

(D) is a fixed line and R is a fixed point that does not belong to (D). A is a variable point on (D). Elongate [RA] to a length $AP = RA$.

1º) Why is point P variable ?

2º) The perpendiculars drawn from R and P to (D) cut it at T and S respectively.

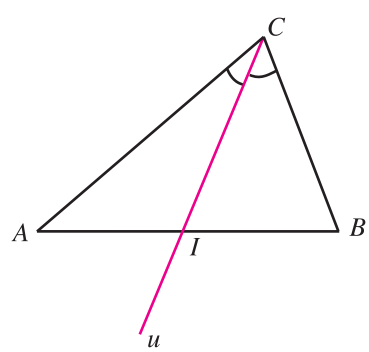
a) Show that triangles RAT and PAS are congruent.

b) Deduce the locus of P.

3 CONSTRUCTIONS

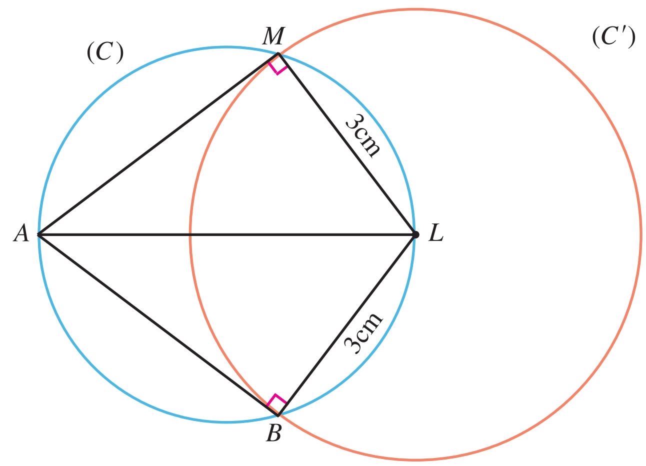
1º) Consider a triangle ABC. On the side [AB], find a point I equidistant from the sides [AC] and [BC].

⊙ Since I is equidistant from the sides [AC] and [BC] of angle $\widehat{ACB}$, then it belongs to the bisector [Cu) of this angle.
Yet I belongs to [AB]; therefore, I is the intersection point of [Cu) and [AB].



2°) [AL] is a fixed segment, such that $AL = 5$ cm.
Construct triangle MAL right at M and such that $LM = 3$ cm.

⊙ Since triangle MAL is right at M, the vertex M is on the circle (C) of diameter AL=5 cm.



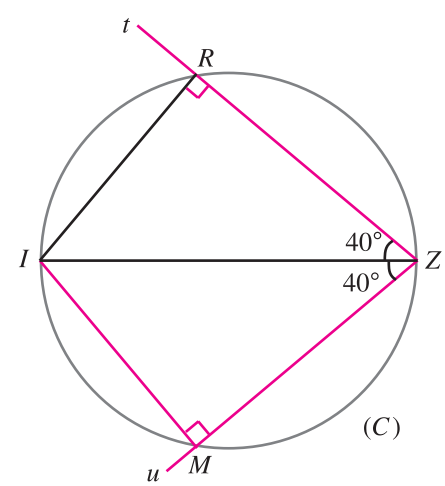
Since $LM = 3$ cm and L is fixed, M is on the circle (C') of center L and radius 3 cm.

Therefore, M is the intersection point of the circles (C) and (C').

The circles (C) and (C') intersect at two points M and B. Therefore, there are two triangles MAL and BAL having the conditions required in the given.

3°) Construct triangle RIZ right at R and such that $IZ = 5$ cm and $\widehat{IZR} = 40°$.

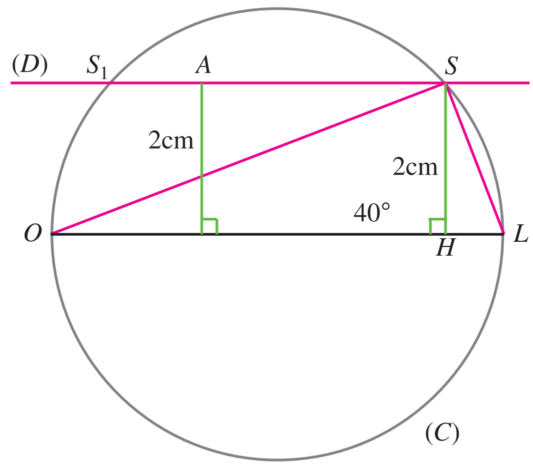
⊙ Since the triangle RIZ is right with a fixed hypotenuse [IZ], the vertex R belongs to the circle (C) of diameter $IZ = 5$ cm.
Since angle $\widehat{IZR}$ is constant and is equal to 40°, the point R belongs to the semi-line [Zt) forming with (IZ) an angle of 40°.



⊙ Therefore, R is the intersection point of (C) and (Zt).
There is a second semi-line [Zu) forming with (IZ) an angle of 40° and that cuts (C) at M.
Therefore, there are two triangles RIZ and MIZ, having the conditions required in the given.

4º) [*OL*] is a fixed segment of length 6 cm and *A* is a given fixed point whose distance from (*OL*) is 2 cm. Construct triangle *SOL* right at *S* with a height [*SH*] measuring 2 cm and such that the vertex *S* is on the same side of *A* with respect to (*OL*).

⊙ Since the triangle *SOL* is right with a fixed hypotenuse [*OL*], the vertex *S* belongs to the circle (*C*) of diameter *OL* = 6 cm.



Since *SH* = 2 cm, then *S* belongs to line (*D*) passing through *A* and parallel to (*OL*).

(*D*) cuts (*C*) at two points *S* and S_1. Therefore there are two triangles *SOL* and S_1OL, having the conditions required in the given.

EXERCISES AND PROBLEMS

Test your knowledge

1 $ABCD$ is a rectangle such that A and B are fixed. The points C and D are variables.

Find the locus of point I center of this rectangle.

2 $MONI$ is a rhombus with M and N fixed, O and I variables.

Find the locus of the points O and I.

3 Consider (C) a circle of center O, diameter $[AB]$ and fixed radius r. M is a variable point on (C). On the ray $[BM)$, we take point N such that $BM = MN$.

1°) Is point O fixed ? Justify.

2°) Show that (OM) is parallel to (AN).

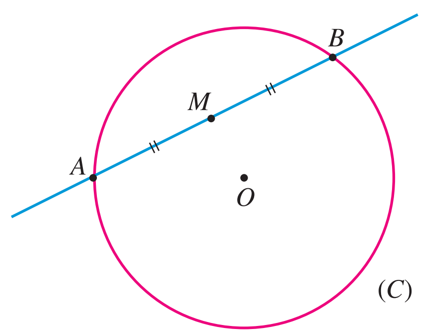
3°) Find the locus of N ?

4 Find and construct the locus of center O of a rhombus $PIED$ such that $[PI]$ is fixed.

5 $[OI]$ is a fixed segment. A variable semi-line $[Ou)$ turns around O. The perpendicular from I to $[Ou)$ cuts it at M.

Find and construct the locus of point M.

6 Consider (C) a fixed circle of center O and A is a fixed point on (C). A variable line, passing through A, cuts again the circle at B. Mark M the midpoint of $[AB]$.

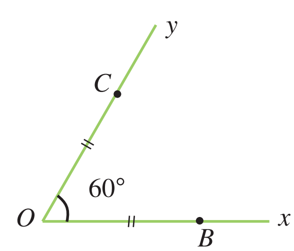


1°) Show that (OM) is perpendicular to (AB).

2°) Find and construct the locus of M.

7 $\widehat{xOy}$ is a fixed angle, such that $\widehat{xOy} = 60°$, B and C are two variable points on $[Ox)$ and $[Oy)$ respectively such that $OB = OC$.

1°) What is the nature of triangle OBC ?



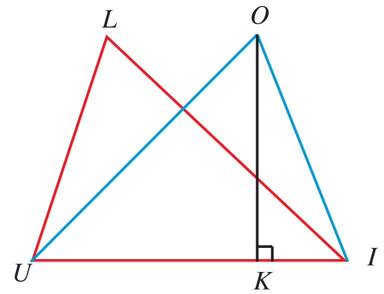
2°) Find and construct the locus of point M, the midpoint of $[BC]$.

8 $\widehat{xOy}$ is a right angle and M is a variable point of $[Ox)$. On the perpendicular at M to $[Ox)$ and inside angle $\widehat{xOy}$, place the point N such that $NM = MO$.

1º) What is the nature of the triangle MON ?

2º) Find and construct the locus of point N.

9 **1º)** OUI is a fixed triangle, such that $UI = 4$ cm. Its height $[OK]$ measures 3 cm. Calculate the area $\mathcal{A}$ of this triangle.



2º) LUI is a triangle with a variable vertex L on the same side of O with respect to (UI) and having the same area as that of the triangle OUI.

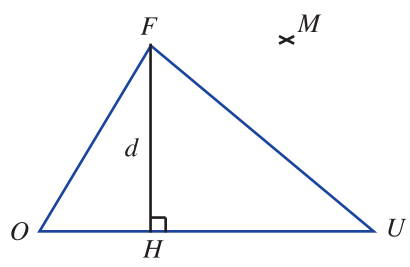
a) Calculate the height $[LH]$ of the triangle LUI.

b) Find and construct the locus of L.

10 $[AS]$ is a fixed segment whose midpoint is I. A point P varies on a fixed line (D) passing through S.

Find and construct the locus of point M, the midpoint of $[PA]$.

11 OUF is a fixed triangle with a height $[FH]$ and $FH = d$. M is a variable point in the plane.



Answer by true or false.

1º) F is a variable point.

2º) $[OU]$ is a fixed segment.

3º) F is at a constant distance from $[OU]$.

4º) If M is equidistant from $[FO]$ and $[FU]$, then M is on the perpendicular bisector of $[OU]$.

5º) If M is on the same side of F with respect to (OU) and at a distance d from $[OU]$, then M varies on the parallel drawn from F to (OU).

6º) If M is equidistant from $[OF]$ and $[OU]$, then M varies on the bisector of $\widehat{FOU}$.

7º) If $\widehat{OUM} = 90°$, then M belongs to the circle of diameter $[OU]$.

8º) If $\widehat{MOU} = 60°$, then M varies on the semi-line $[Ot)$ forming with (OU) an angle of 60°.

9º) If $\widehat{FMU} = 90°$, then M belongs to the circle of diameter $[FU]$.

10º) If (MH) is perpendicular to (OU), then M varies on the line (FH).

For seeking

12 A and B are two fixed points. (C) is a variable circle of center O passing through A and B.

What is the locus of O ?

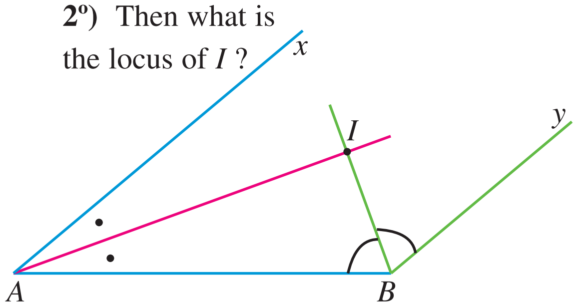
13 ABC is an isosceles triangle of variable vertex A and a fixed base $[BC]$.

The bisectors of the angles $\widehat{ABC}$ and $\widehat{ACB}$ intersect at I.

1°) I is a variable point. Justify.

2°) What is the locus of I ?

14 A and B are fixed.
$[Ax)$ and $[By)$ are variables and parallels.
The bisectors of the angles $\widehat{BAx}$ and $\widehat{ABy}$ intersect at I.
1°) What is the nature of the triangle ABI ?
2°) Then what is the locus of I ?



15 Consider a fixed angle $\widehat{xAy} = 60°$. M is a variable point of $[Ax)$ and N is the symmetric of M with respect to $[Ay)$.

1°) What is the nature of triangle AMN ?

2°) Find and construct the locus of N.

16 $\widehat{xOy}$ is a fixed angle. A is a variable point on $[Ox)$. From A draw the parallel to $[Oy)$. On this parallel, place the point B such that $AB = AO$ $\big([AB)$ and $[Oy)$ are on the same side with respect to $[Ox)\big)$.

Find and construct the locus of point B.

17 EST is a triangle, such that E and S are fixed points and T is a variable point with $\widehat{EST} = 50°$. I is the midpoint of $[SE]$.

1°) How does point T move ?

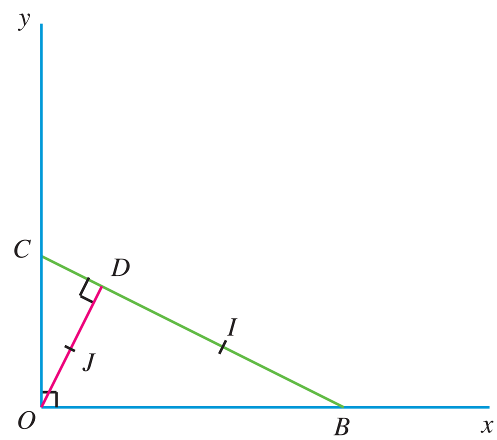
2°) Find and construct the locus of M, the midpoint of $[ET]$.

18 A is a variable point on a semi-circle of fixed diameter $[BC]$. On the semi-line $[BA)$, locate a point D, such that $BD = AC$. On the semi-line $[Bx)$, the tangent at B to this semi-circle, place a point E, such that $BE = BC$.

1°) Show that the triangles ABC and BDE are congruent. Deduce the measure of angle $\widehat{BDE}$.

2°) Find and construct the locus of point D.

19 Consider two perpendicular semi-lines $[Ox)$ and $[Oy)$. B is a variable point of $[Ox)$ and C is a fixed point of $[Oy)$.
The perpendicular from O to (BC) cuts it at D.
I and J are respectively the midpoints of $[BD]$ and $[OD]$.



1º) Show that the line (IJ) is perpendicular to the line (OC).

2º) What does point J represent for triangle OIC ?

3º) Consider F, the intersection point of the lines (OI) and (CJ). Find and construct the locus of point F when B varies on $[Ox)$.

Constructions

20 Construct a triangle SOL right at S, such that $OL = 8$ cm and $SO = 5$ cm.

21 O and I are two fixed points, such that $OI = 6$ cm. Construct a right isosceles triangle with hypotenuse $[OI]$.

22 SOL is a triangle.

Construct a point I equidistant from $[SO]$ and $[OL]$ such that $IS = IO$.

23 Draw a fixed segment $[IL]$ of length 8 cm. K is a point of $[IL]$ such that $IK = 5$ cm. On the semi-line $[Kx)$ perpendicular at K to (IL), place point E, such that $KE = 2$ cm.

Construct a triangle FIL right at F and whose height $[FH]$ measures 2 cm.

24 A and S are two fixed points, such that $SA = 3$ cm.

Construct a triangle PAS right at S and such that $AP = 4$ cm.

25 Construct an equilateral triangle MAL with height $[MH]$, such that $MH = 2$ cm.

TEST

1 Complete the following table (M is a variable point of the plane).

Property of M	Locus of M
	M belongs to the bisector $[Ou)$ of angle $\widehat{xOy}$
$\widehat{AMB} = 90°$ (A and B are fixed)	
$\widehat{MAB} = 50°$ (A and B are fixed)	
	M forms with the fixed point A a line (D) parallel to the fixed line (xy)

(4 points)

2 Consider a triangle SAL, such that L and S are fixed points and A is a variable point.

Find and construct the locus of A if $\widehat{ASL} + \widehat{LAS} = 130°$. **(2 points)**

3 Consider two fixed points A and T; F is a fixed point of segment $[AT]$. On the perpendicular (xy) drawn from F to (AT), plot a variable point R. The perpendicular from T to (AR) cuts (xy) at I and (AR) at J.

1º) What does point I represent for triangle RAT ? **(2 points)**

2º) Find and construct the locus of M, the intersection point of (AI) and (RT).

(3 points)

4 (xy) is a fixed line and A is a fixed point that does not belong to (xy); $[AH]$ is the perpendicular from A to (xy) and I is the midpoint of $[AH]$.

1º) Why is point I fixed ? **(2 points)**

2º) A variable line, passing through A, cuts (xy) at B. Find and construct the locus of M, the midpoint of $[AB]$. **(3 points)**

5 TAS is a triangle. In the interior of this triangle, construct the point I equidistant from the three sides of the triangle. **(4 points)**

24 SOLID GEOMETRY (SPACE GEOMETRY)

Objective

To know the relative positions of two straight lines, of two planes, and of a straight line and a plane.

CHAPTER PLAN

COURSE

1 PLANE

The triangle ABC is in a plane called plane of this triangle, sometimes denoted by (P) or (Q) or (R)...

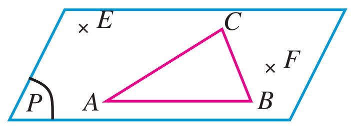


Fig. 1

The plane is represented by a parallelogram.

The plane is unlimited.

2 DETERMINATION OF A PLANE

Three non collinear points A, B and C determine one and **only one plane** denoted by (ABC).

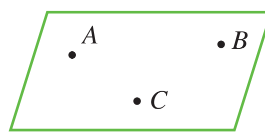


Fig. 2

Two intersecting lines (D) and (D') determine **one** and **only one plane** denoted by $\big((D), (D')\big)$.

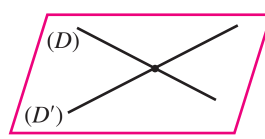


Fig. 3

Two parallel lines (EF) and (BC) determine **one** and **only one plane** denoted by $\big((EF), (BC)\big)$.

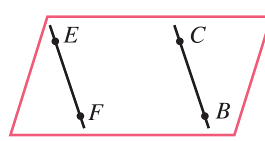


Fig. 4

A line (D) and a point C that does not belong to (D), determine **one** and **only one plane** denoted by $\big((D), C\big)$.

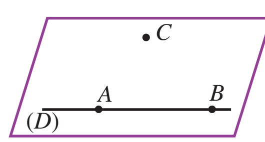


Fig. 5

3 RELATIVE POSITION OF TWO PLANES

ABCDEFGH is a right prism.

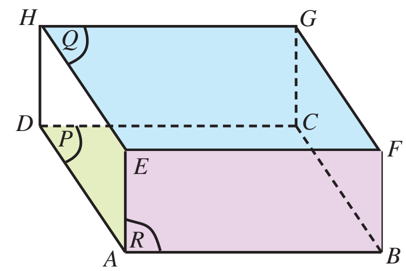


Fig. 6

1º) Parallel planes

The faces *ABCD* and *EFGH* are respectively the planes (*P*) and (*Q*). These planes do not have a common point. They are called **parallel planes** and **(*P*) // (*Q*)**.

Two planes are parallel if they have no point in common.

2º) Intersecting planes

The face *ABFE* is in the plane (*R*).

The planes (*P*) and (*R*) have only line (*AB*) in common.

The planes (*P*) and (*R*) are called **two intersecting planes and their intersection** (common part) **is the line (*AB*).**

Two intersecting planes intersect along a line.

Application 1

The faces of the right prism of figure 3 are planes. Name these planes.

Among these planes, which are parallel ?

Among the planes of the faces, name those that are intersecting and determine their intersection two by two.

4 RELATIVE POSITION OF TWO LINES

ABCDEFGH is a right prism. (fig 4)

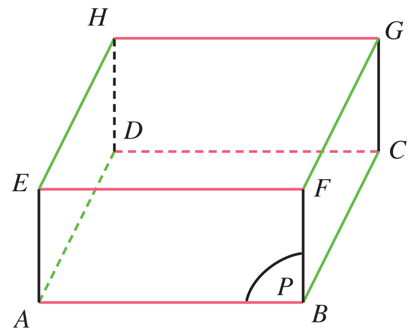


Fig. 7

1º) Coplanar lines

The quadrilateral *ABFE* is a rectangle in plane (*P*).

In this plane (*P*) the lines (*AB*) and (*EF*) are parallel and the lines (*AB*) and (*AE*) are intersecting.

Two lines of a plane are called coplanar if they are parallel or intersecting.

In plane (*P*) :

- ⊙ (*AB*) // (*EF*),
- ⊙ (*AB*) and (*AE*) are intersecting lines.

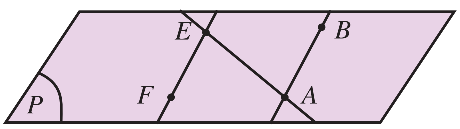


Fig. 8

2º) Non-coplanar lines

The lines (*AB*) and (*EH*) of figure 4 are not in the same plane :

(*AB*) and (*EH*) are neither parallel nor intersecting, they are called : **two non-coplanar lines** (skew lines)**.**

Remark

Two lines, which are not intersecting, are not necessarily parallel.

(*AB*) and (*FG*) of figure 4 do not intersect and are not parallel.

Application 2

Name the parallel lines in figure 7.

Name four pairs of intersecting lines in figure 7.

Can we conclude that : **two lines parallel to a third line are parallel ?**

5 RELATIVE POSITIONS OF A LINE AND OF A PLANE

(*P*) is the plane of the base *ABCD* of a prism *ABCDEFGH* (fig. 8).

1º) Line in a plane

The line (*AB*) has two points *A* and *B* in the plane (*P*).
The line is in (*P*).

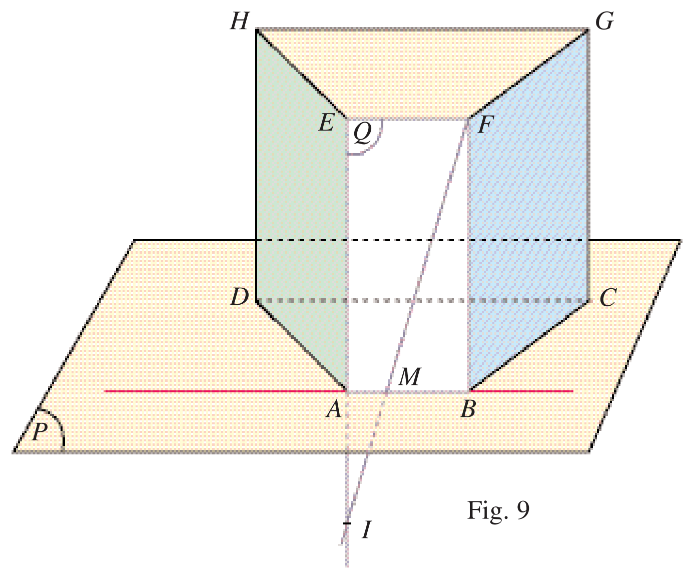


Fig. 9

Any line having two points in a plane lies completely in this plane.

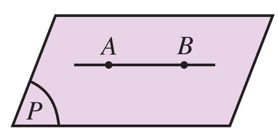


Fig. 10

(*AB*) is in (*P*).

2º) Line intersecting a plane

F is not a point in (*P*) and *M* is a point of the line (*AB*) (fig. 8)

The line (*FM*) is not in (*P*) because the point *F* is not in (*P*). The line (*FM*) cuts (*P*) at *M*.

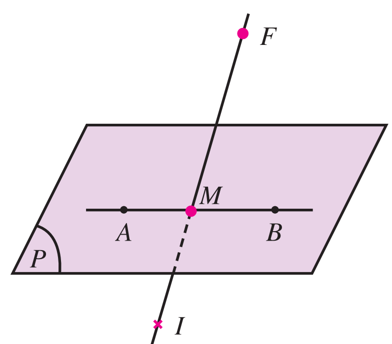


Fig. 11

(*FM*) cuts (*P*).

A line having only one common point with a plane is intersecting a plane. This line cuts or intersects (*P*). (fig. 10)

3º) Line parallel to a plane

The line (*EF*) has no common point with plane (*P*) (fig. 8).
The line (*D*) is parallel to plane (*P*).

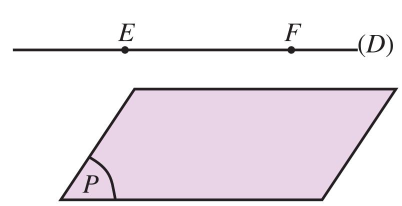


Fig. 12

A line (*D*) having no common point with a plane (*P*) is called parallel to (*P*) : (*D*) // (*P*). (fig. 11)

EXERCISES AND PROBLEMS

Test your knowledge

1 Answer by true or false.

1º) A line, having two points in a plane, lies in this plane.

2º) Two non-coplanar lines intersect in one point.

3º) Two lines that do not intersect are parallel.

4º) Two intersecting planes intersect along a line.

5º) Two parallel planes have one common line.

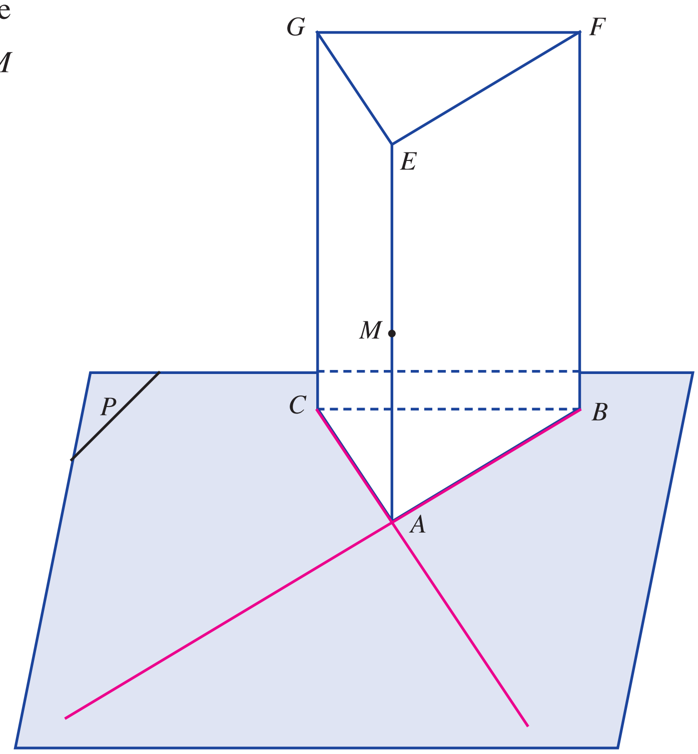
6º) A line intersecting a plane cuts it at two distinct points.

2 Consider (P) a plane of the base ABC of a prism $ABCEFG$ and M the midpoint of the edge $[AE]$.

1º) Is (EF) parallel to (AB) ?
Is (FM) parallel to (AB)?

2º) (FM) cuts (AB) at I.
Is point I a point of (P) ?

3º) Line (GM) cuts (P) at J.
Place the point J.

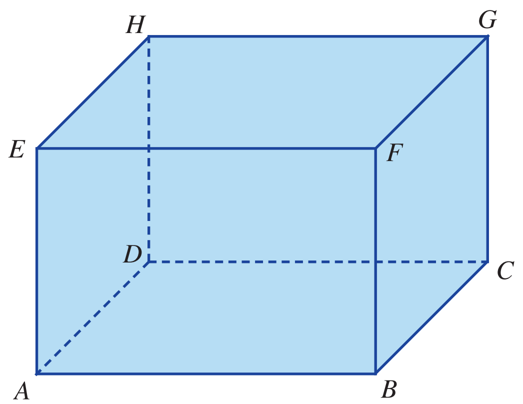


3 *ABCDEFGH* is a right prism.

1º) Are the lines (AE) and (CG) coplanar ? Why ?

2º) Are the lines (BF) and (DH) coplanar ? Why ?

3º) Consider I and J the centers of the rectangles $ABCD$ and $EFGH$. Draw the intersection of the planes $\big((AE)\,,(CG)\big)$ and $\big((BF)\,,(DH)\big)$.

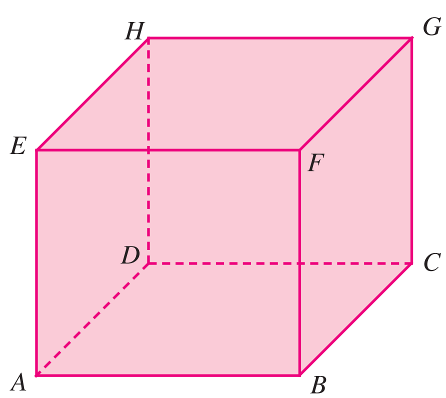


4 Consider O and O' the centers of the bases $ABCD$ and $EFGH$ of a cube $ABCDEFGH$.

1º) Show that the quadrilateral $ACGE$ is a parallelogram.

2º) Show that (OO') is parallel to (AE).

3º) Show that (OO') lies in the plane formed by the lines (BF) and (DH).

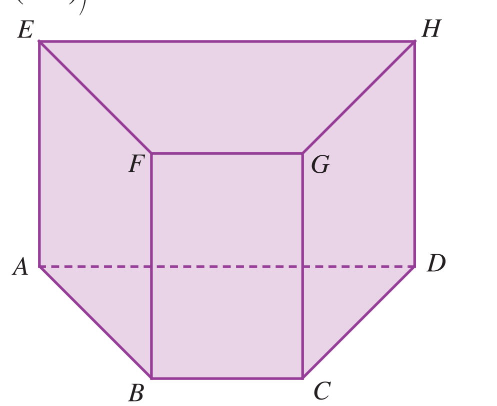


For seeking

5 The trapezoids $ABCD$ and $EFGH$ are the bases of a prism. Consider I the intersection point of lines (AB) and (DC).

1º) Is I a point of the plane having $ABCD$ as base ?

2º) Consider J the intersection point of (EF) and (GH). Draw the intersection of planes $\big((AB), (FE)\big)$ and $\big((CD), (GH)\big)$.

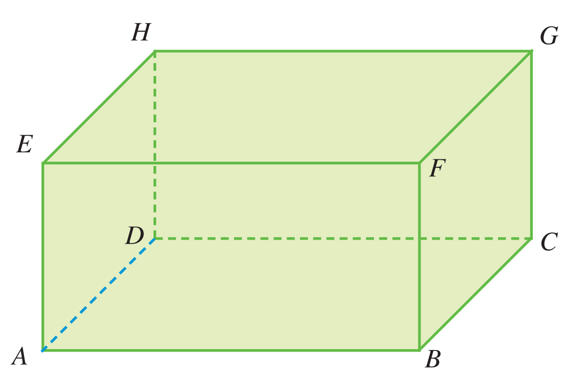


6 *ABCDEFGH* is a right prism.

1º) Show that $AHGB$ is a parallelogram.

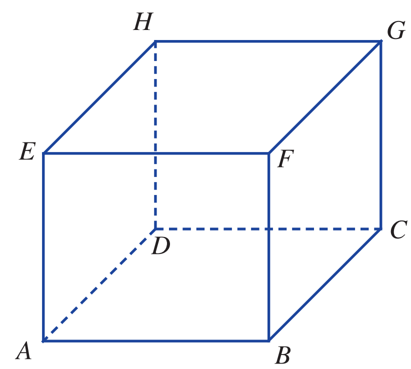
2º) Are (HG) and (AB) parallel ?

3º) Draw the intersection of planes $\big((AB), (GH)\big)$ and $\big((DH), (BF)\big)$.



TEST

1 Consider M, N, P and Q the respective midpoints of the edges $[AD]$, $[EH]$, $[AB]$ and $[EF]$ in a cube $ABCDEFGH$.



1º) Draw the plane containing the points M, N, P and Q. **(2 points)**

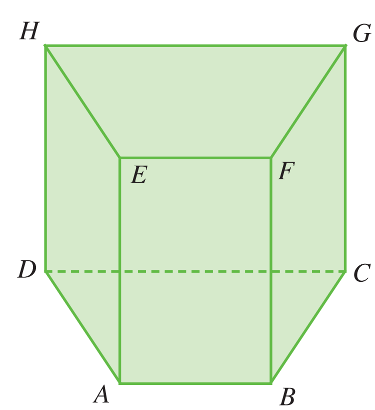
2º) Are (MP) and (BD) parallel ? **(2 points)**

3º) Are (MN) and (HG) coplanar ? **(2 points)**

4º) Name two parallel planes. **(2 points)**

5º) Name the intersections of plane $\big((MN), (PQ)\big)$ with planes $\big((AB), (CD)\big)$, $\big((AD), (HE)\big)$ and $\big((EF), (GH)\big)$. **(3 points)**

2 $ABCDEFGH$ is a prism whose bases $ABCD$ and $EFGH$ are isosceles trapezoids.



1º) Name the vertices of the prism that are in the plane (ACE). **(3 points)**

2º) What is the plane, in the figure, that is parallel to the plane $\big((AE), (BF)\big)$? **(3 points)**

3º) Are the points C, D, E and F in the same plane ? **(3 points)**

25 PYRAMID

Objectives

1. To draw a pyramid with a given base.

2. To calculate the surface area and the volume of a pyramid.

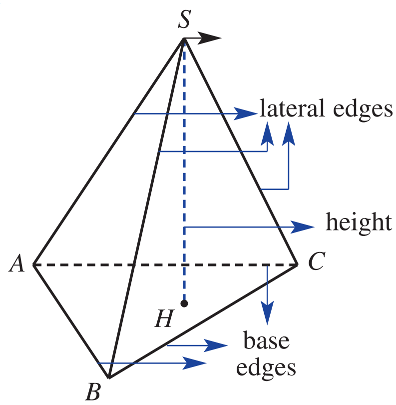
CHAPTER PLAN

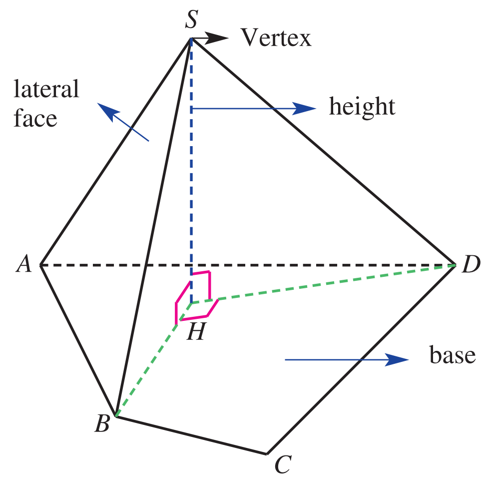
COURSE

EXERCISES AND PROBLEMS

TEST

1 DEFINITIONS





A **pyramid** is a solid (polyhedron) limited by :

A **polygonal base** (triangle, quadrilateral, pentagon, ...) and **triangular lateral faces.**

The vertex of the pyramid is **the vertex common to all the lateral faces.**

[*SH*] is the **height** of the pyramid.

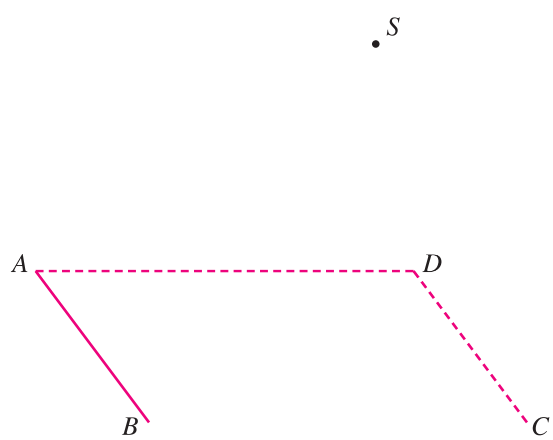
Remark

The number of lateral faces in a pyramid is equal to the number of its base edges.

Application 1

S is the vertex of the pyramid whose base is the parallelogram *ABCD*.

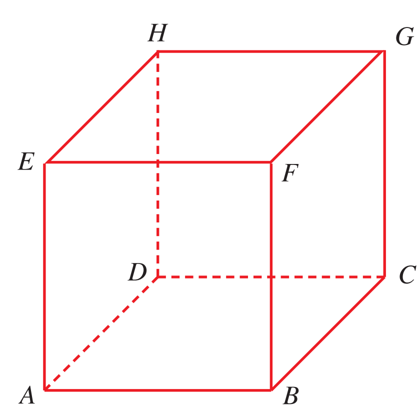
Complete this pyramid.



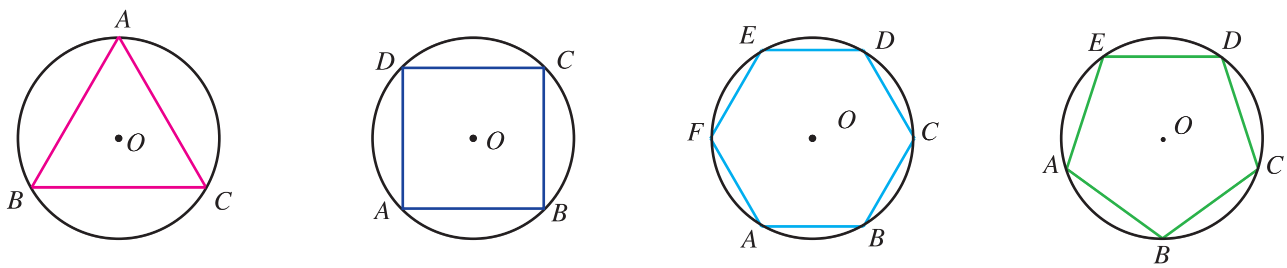
Application 2

ABCDEFGH is a cube. Draw the pyramid of vertex *E* and base *ABD*.

Is the segment [*EA*] the height of this pyramid ? Justify.



2 REGULAR POLYGON



A polygon is regular if all its sides are congruent and all its angles are equal.

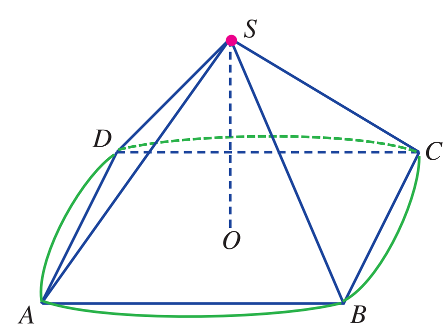
Remarks

⊙ Any regular polygon is inscribed in a circle.

⊙ The equilateral triangle and the square are regular polygons.

⊙ The center O of the circle circumscribed about the regular polygon is the center of this polygon.

3 REGULAR PYRAMID



A pyramid is regular if the base is a regular polygon and its faces are isosceles triangles.

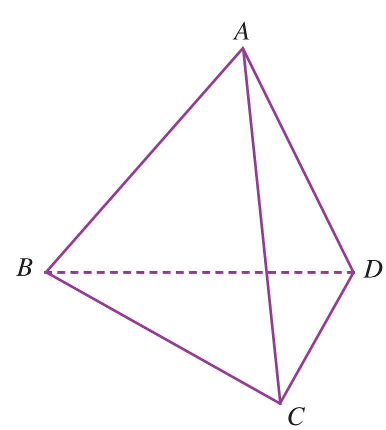
Remarks

In a regular pyramid :

⊙ the faces are congruent isosceles triangles.

⊙ the segment from the vertex S to the center O of the base is the altitude-segment.

4 TETRAHEDRON



A tetrahedron is a pyramid with a triangular base.

A tetrahedron is regular if its four faces are equilateral triangles.

5 AREA OF A PYRAMID

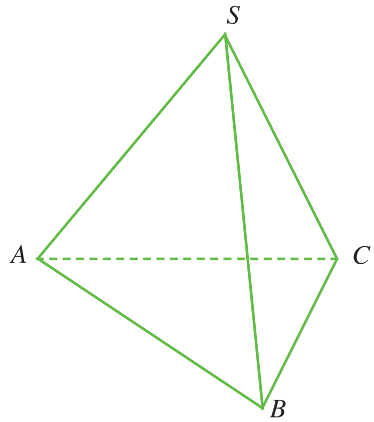
The lateral area $\mathscr{A}_\ell$ of a pyramid is equal to the sum of the areas of the lateral faces.
The total area (surface area) $\mathscr{A}_t$ of a pyramid is equal to the sum of its lateral area and its base area.

Application 3

SABC is a regular tetrahedron with edge $SA = 6$ cm.

Verify that the lateral area of this tetrahedron is the triple of the area of triangle *SAB*.

Verify that the total area is $36\sqrt{3}$ cm².



6 VOLUME OF A PYRAMID

If V is the volume of a pyramid, $\boxed{V = \dfrac{\mathscr{B} \times h}{3}}$ where $\mathscr{B}$ is the area of the base and h is the height.

Application 4

Calculate the height of a pyramid having a volume of 64 cm³ and whose base is a square of side 4 cm.

Exercises and problems

Test your knowledge

1 Answer by true or false.

1º) Any right prism is a pyramid.

2º) The number of edges from the vertex of a pyramid is equal to the number of its lateral faces.

3º) If the lateral edges of a pyramid have the same length, the pyramid is regular.

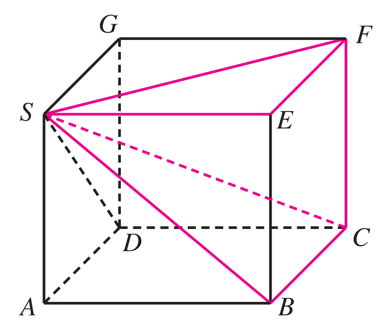
4º) A regular pyramid can have a rectangle as a base.

5º) The lateral faces of a regular pyramid are always equilateral triangles.

6º) The bases of two pyramids, having the same volume and the same height, are congruent.

7º) The bases of two pyramids, having the same volume and the same height, have the same lateral area.

2 *ABCDSEFG* is a cube.



1º) What is the vertex of the pyramid *SABCD* ?

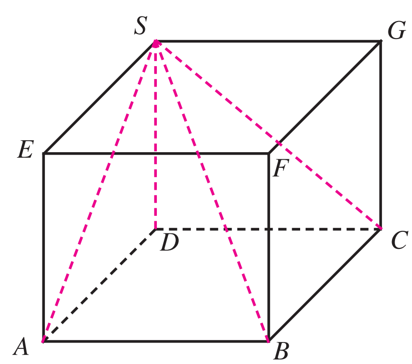
2º) Name the lateral faces of this pyramid.

3º) What is the height of this pyramid ?

3 *ABCDEFGH* is a right prism. Draw two pyramids with the same vertex *H* and with respective bases the rectangles *ABFE* and *ABCD*.

What is the common face of these two pyramids ?

4 *ABCDEFGS* is a cube with edge $AB = 4$ cm.
S is the vertex of the pyramid with base *ABCD*.



1º) Determine two pairs of congruent faces in this pyramid.

2º) Is [*SD*] the altitude-segment of this pyramid ?

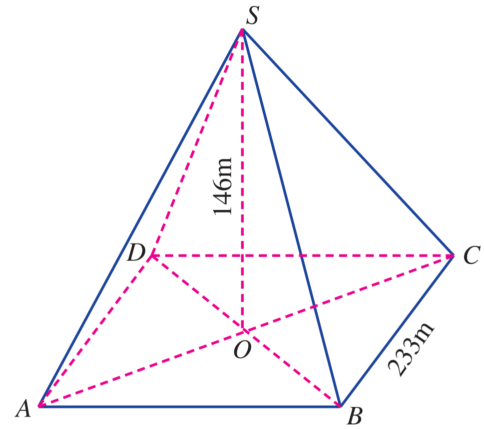
3º) Calculate the lateral area and the volume of this pyramid.

For seeking

5 The pyramid of Cheops is a regular pyramid with a square base of 233 m the side and 146 m the height.

By using the pythagorean theorem , show that the edges $[SA]$, $[SB]$, $[SC]$ and $[SD]$ have the same length.

Calculate this length.

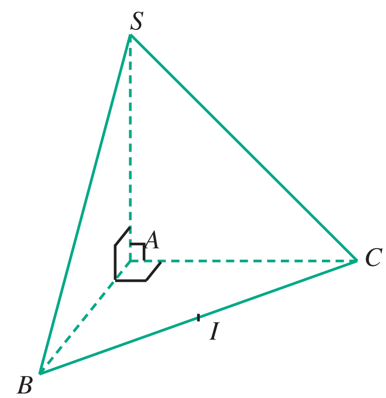


6 The base ABC of the pyramid $SABC$ is a right isosceles triangle at A with $AB = 4$ cm. $[SA]$ is the altitude-segment of this pyramid and $SA = 4$ cm.

1º) Calculate SB, SC and BC.

2º) I is the midpoint of $[BC]$; Calculate SI.

3º) Calculate the total area $\mathscr{A}_t$ and the volume V of this pyramid.



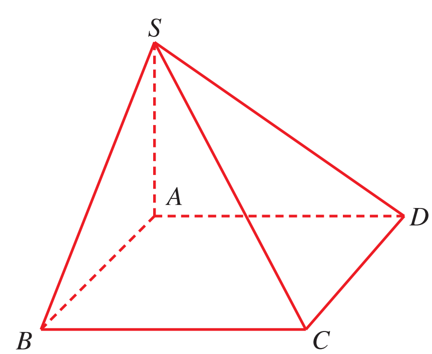
TEST

1 $SABCD$ is a pyramid with a rectangular base and whose altitude-segment is $[SA]$.

1º) Calculate the volume of this pyramid if $AB = 4$ cm, $AD = 9$ cm and $SA = 5$ cm. **(5 points)**

2º) I, J, K and L are respectively the midpoints of $[SA]$, $[SB]$, $[SC]$ and $[SD]$.

What is the vertex of the pyramid $SIJKL$? What is the nature of its base ? **(5 points)**



26 CYLINDER - CONE - SPHERE

Objectives

1. To draw a cylinder, a circular cone, a sphere.
2. To calculate the volume of a cylinder, the volume of a circular cone, and the volume of a sphere.

CHAPTER PLAN

COURSE

1. Circular Cylinder
2. Circular Cone
3. Sphere and ball

EXERCISES AND PROBLEMS

TEST

1 CYLINDER

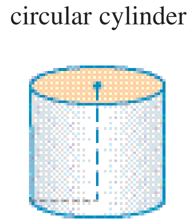


Fig. 1

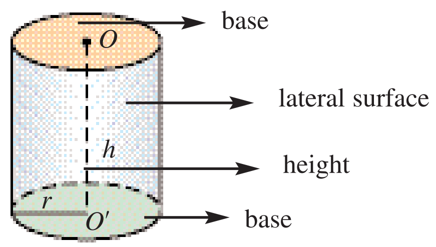


Fig. 2

A cylinder is limited by two congruent discs called circular bases that lie in parallel planes. O and O' are the centers of the discs. $[OO']$ is the altitude.

The lateral area $\mathscr{A}_\ell$ of a circular cylinder is :

$\boxed{\mathscr{A}_\ell = 2\,\pi r h}$ where r is the radius of the base and h is the altitude of the cylinder.

The total area (surface area) $\mathscr{A}_t$ of a cylinder is the sum of the lateral area and the areas of the bases.

$\boxed{\mathscr{A}_t = 2\,\pi r h + 2\,\pi r^2}$

The volume V of a circular cylinder is :

$\boxed{V = \pi r^2 h}$

2 CIRCULAR CONE

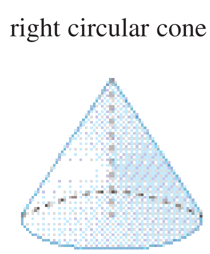


Fig. 3

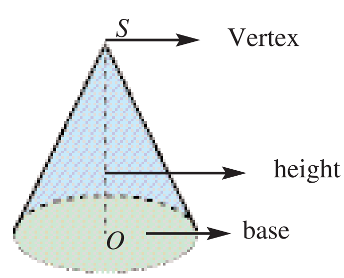


Fig. 4

The base of a right circular cone is a disc.
$[SO]$ is the altitude of the right circular cone, O is the center of the base and S is the vertex of the cone.
The volume V of a circular cone is :

$\boxed{V = \dfrac{\pi r^2 h}{3}}$ where r is the radius of the base and h is the height of the cone.

3 SPHERE AND BALL

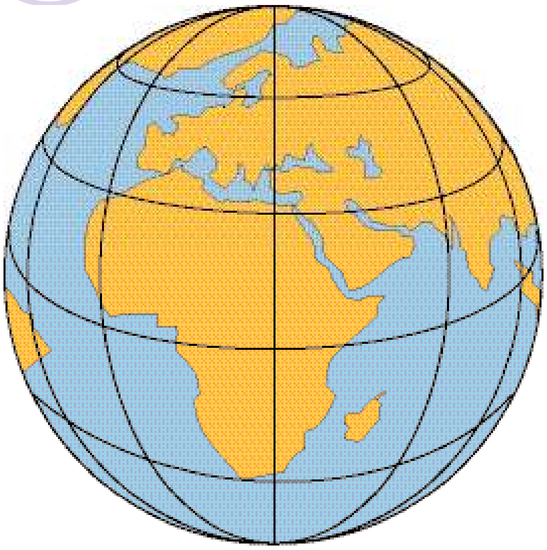


Fig. 5

sphere

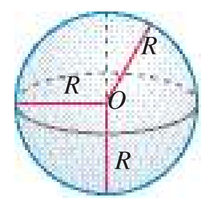


Fig. 6

The sphere of center O and radius R is the set of points in space whose distance to O is equal to R.

This sphere is denoted by $S\,(O, R)$.

The ball of center O and radius R is the solid limited by the sphere $S\,(O, R)$.

This ball is denoted by $B\,(O, R)$.

The volume V of the ball with center O and radius R is :

$$\boxed{V = \frac{4\pi R^3}{3}}$$

EXERCISES AND PROBLEMS

Test your knowledge

1 Answer by true or false.

1°) The base of a cylinder is a disc.

2°) The two bases of a cylinder have the same radius.

3°) A circular cone and a cylinder having the same height and same base, have the same volume.

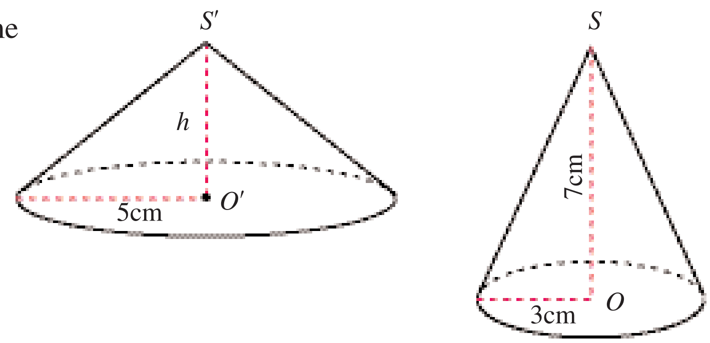
4°) The volume of a ball $B\,(O, R)$ is half the volume of a ball $B\,(O, 2R)$.

2 Calculate the lateral area $\mathcal{A}_\ell$ of a cylinder with a height 10 cm and whose base radius is 20 cm.

3 Calculate the radius of a ball with volume 113.04 cm^3 ($\pi = 3.14$).

4 Two circular cones have the same volume.

Calculate h.



5 What is the area of the base of a cylinder with height 1.2 m and volume 8.4 m^3 ?

6 A can closed from both extremities, has the form of a cylinder with radius 0.6 m and height 1.35 m.

1º) Calculate its volume in liters ($1\ \ell = 1\ \text{dm}^3$).

2º) Calculate the area of the surface of the metal sheet used to make this can.

7 A glass full of water has the form of a cylinder with diameter 5 cm and height 8 cm.

1º) Calculate the volume of water.

2º) A steel marble of diameter 3 cm is immersed in the glass.

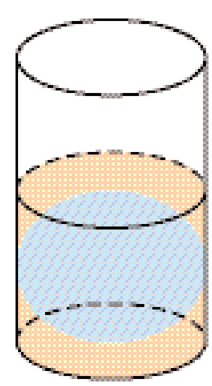
What is the quantity of water that overflows ?

For seeking

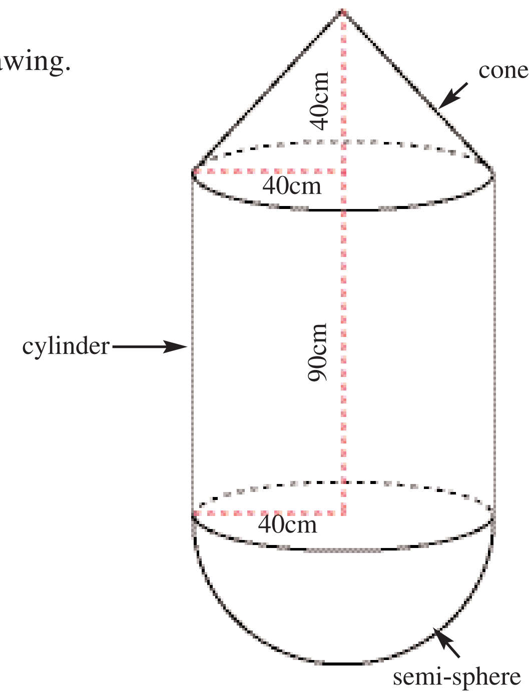
8 A spherical ice cube of radius 3 cm is immersed in a cylindrical glass of radius 3 cm, containing water.

The ice cube lies at the bottom of the glass, and water exactly covers the cube.

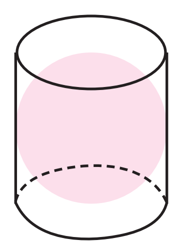
What was the height of the water in the glass ?



9 Calculate the volume corresponding to this drawing.



10 A spherical balloon having a diameter 20 cm is placed in a cylinder of height 20 cm and base diameter 20 cm.



Calculate :

1º) The lateral area $\mathcal{A}$ of the cylinder.

2º) The area $\mathcal{A}'$ of the balloon.

3º) Compare both areas.

4º) Calculate :

a) The volume $\mathcal{V}$ of the cylinder.

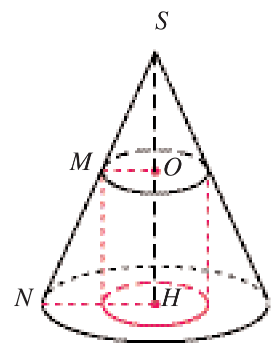
b) The volume $\mathcal{V}'$ of the balloon.

c) The quotient $\dfrac{\mathcal{V}}{\mathcal{V}'}$.

TEST

1 In this figure, $[SH]$ is the altitude of a circular cone; $[OH]$ is the altitude of a cylinder.

Suppose that $SH = 7$ cm, $HN = 3$ cm and $SO = 3.5$ cm.

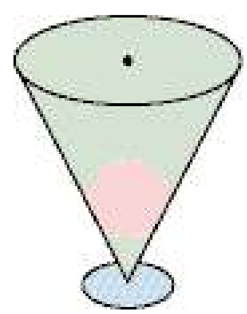


1º) Calculate OM, the radius of the base of the cylinder. **(2 points)**

2º) Calculate the volume of the cone and the volume of the cylinder. **(3 points)**

2 A glass full of water has the form of a circular cone with height 7 cm and radius 3 cm.

A steel ball with radius 2 cm is immersed in the glass.



1º) What is the quantity of water that overflows ? **(2 points)**

2º) What is the quantity of water that remains in the glass ? **(3 points)**